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A hybrid Monte Carlo acceleration method of pricing basket options based on splitting

Yongchao Sun [a], Chenglong Xu [b,*]

[a] *School of Mathematical Sciences, Tongji University, Shanghai, 200092, China*
[b] *School of Mathematics, Shanghai University of Finance and Economics, Shanghai, 200433, China*

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ABSTRACT

Pricing basket options has always been one of the key problems in financial engineering because of high dimensionality and low convergence rate. This paper proposes a hybrid Monte Carlo variance reduction method for pricing basket options. First, by splitting the payoff of the basket option into two parts, we can price basket options by value the two parts respectively. The first part has a closed-form expectation formula, the second part can be considered as a small probability event. To reduce variance for simulating the second part, the conditional Monte Carlo (CMC) method combined with the importance sampling(IS) method is adapted. Because these two methods are all effective to deal with small probability events. For IS method, it is a challenge to compute the optimal parameters with as little computational cost as possible. Therefore, an efficient prediction–correction(PC) iteration algorithm based on moment estimation is proposed to determine the optimal parameters in the importance sampling method. Some theoretical analyses for the existence and uniqueness of the optimal parameters in the IS method and the convergence of the PC method are also given. Numerical results show that the hybrid variance reduction method has great variance reduction effect and PC iteration algorithm can save a lot of computing costs comparing with the traditional Newton's iteration method.

1. Introduction

Basket options are popular in financial market. The payoff of basket options is correlated with the arithmetic average prices or weighted arithmetic average prices of several assets at expired time. The basket options are widely used in index trading and foreign-exchange market. However they do not have a closed-form price formula, so many researching works have been done about pricing basket options by numerical methods, i.e. tree method, Fourier transform method and methods based on partial differential equation (PDE) like the finite element methods (FEM) and the finite difference methods (FDM). Topper [1] used FEM to price many kinds of exotic options including basket options. Lötstedt, Persson, Von Sydow, et al. [2] used the FDM to price basket options. Borovkova, Permana and Weide [3] used the binomial tree method to price and hedge basket options. Leentvaar and Oosterlee [4] used the fast Fourier transform (FFT) method to price basket options. Larsson, Ahlander and Hall [5] used general Fourier transform (GFT) method to price basket options. Jiang, Liu and Yu [6] priced Asian basket options by FEM. For these numerical methods, one problem is that they cannot price options with too many assets otherwise the computation cost would be very large. However in practice, the number of assets in basket options would be very large, especially in some index trading.

* Corresponding author.
E-mail address: clxu@tongji.edu.cn (C. Xu).

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Monte Carlo method is a useful numerical method for option pricing. For low-dimensional problems, the efficiency of Monte Carlo method may be not as good as some other numerical methods because the convergence rate is only $O(m^{-\frac{1}{2}})$, m is the sample numbers. But Monte Carlo method exhibits flexibility for high-dimensional problems, since its convergence rate does not depend on the dimension of the problems.

In practice, some variance reduction methods are used for acceleration the simulation. Kemna and Vorst [7] first used the payoff of geometric basket options as a control variate to reduce variance for pricing basket options. They used the fact that the payoff of arithmetic averaged basket options and geometric averaged basket options are highly related and the price of geometric average basket options has a closed-form formula. Pellizzari [8] used several one-asset options as the control variates to price basket options. Giles [9] used multilevel Monte Carlo method in pricing basket options. Glasserman, Heidelberger and Shahabuddin [10] proposed an importance sampling method for pricing basket options. Joy, Boyle and Tan [11] and Petroni and Sabino [12] used Quasi Monte Carlo method in basket option pricing and hedging. Hu and Chen [13] also used Quasi Monte Carlo method, where parallel computation was used in Monte Carlo simulations. Korn and Zeytun [14] improved the efficiency of variance reduction effect of Korn and Zeytun [15] by adding a constant on the payoff of geometric average basket option. Dingeç and Hörmann [16] combined the methods of control variate and conditional Monte Carlo method together, for pricing basket options and Asian options.

To determine the optimal parameters in IS method, Glasserman, Heidelberger and Shahabuddin [10] used large deviations theory and got asymptotically optimal measure for IS method. Jiang, Xu and Fu [17] used sample average approximation (SAA) method and the Newton's iteration method to solve the corresponding optimization problem related to the IS method. In fact, as a traditional method, the Newton's iteration method is very effective in solving the optimization problem with smooth target function, although extra time is required to compute the inverse of the Hessian matrix at every iteration step, especially in the high-dimensional cases. We will also prove the convergence and give the convergence rate theoretically of the Newton's iteration method for our problem. The Newton's iteration method is also compared with our PC algorithm to show the effectiveness and efficiency in this paper.

Besides financial problems, optimization methods also have many other applications in other areas. For example, Zhang and Chau [18] proposed a method to solve plant leaf image classification problem. They used semi-supervised locally linear embedding (SLLE) method with difference parameters and then got the optimal parameters. Some problems in rainfall prediction are also correlated with optimization methods, see Chau and Wu [19] and Taormina and Chau [20]. Hong, Doumith and Davison [21] used machine learning method to structure recommender systems for social networks. Optimization problems are also useful in civil engineering area, see Rackwitz and Joanni [22]. Some other applications of optimization methods in different areas can be seen in [23–25].

In this paper, we construct a hybrid variance reduction method, which combines the control variate (CV) method, conditional Monte Carlo (CMC) and importance sampling (IS) method altogether to get better variance reduction effect. Furthermore, the PC algorithm we proposed in this paper can save a lot computational cost for computing the optimal parameters in IS method.

There are also many works for estimating the price of basket option by other methods. Curran [26] estimated arithmetic averaged Asian option's price by geometric average Asian option. Milevsky and Posner [27] used Gamma distribution to estimate the price of a basket option under the assumption that the average value of the asset's prices is under Gamma distribution. Leccadito, Paletta and Tunaru [28] used exact moment matching idea to estimate the price of basket options. Caldana, Fusai and Gnoatto, et al. [29] estimated the bound of basket options. In this paper, some idea in [26] is used to split the payoff of the basket options into two parts, which can be considered as a CV method to reduce variance of simulation. This is the first step in our hybrid variance reduction method.

The rest of the paper is organized as follows. Section 2 gives a brief introduction of basket options and Monte Carlo method of basket options. Section 3 proposes our new hybrid variance reduction method for basket options, some essential formulas and theory analysis are derived. Section 4 gives numerical results and Section 5 concludes the paper.

2. Monte Carlo simulation for basket option

A basket option is a derivative written on multiple assets, whose payoff depends on the average or weighted average price of several underlying assets at expired time. Consider a basket option written on n underlying assets with payoff

$$(\sum_{i=1}^{n} w_i S_{iT} - K)^+,$$

where $S_{it}(i = 1, 2, \dots, n)$ are the prices of the assets at time t, T is the maturity, K is the strike price, $w_i(i = 1, 2, \dots, n)$ are the weights of the assets. Assume the assets are under Geometric Brownian Motion (GBM), i.e.

$$S_{it} = S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{t}x_i},$$

where r is the risk free interest rate, σ_i are volatilities of the assets and δ_i are dividend fields, x_i are corresponding standard normal distributions. Let

$$A = \sum_{i=1}^{n} w_i S_{iT} = \sum_{i=1}^{n} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{T}x_i},$$

then the price of the basket option at initial time under risk neutral measure is

$$V = e^{-rT}E[(A-K)^+].$$

However, the closed-form of the basket option price does not exist, so some numerical methods are needed to price basket options.

Monte Carlo simulation is a common used numerical method in financial engineering, it uses average value of samples to estimate expectation value. A basket option price can be approximated by

$$\hat{V} = e^{-rT}\frac{1}{m}\sum_{j=1}^{m}(A^{(j)}-K)^+ = e^{-rT}\frac{1}{m}\sum_{j=1}^{m}(\sum_{i=1}^{n} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{T}x_i^{(j)}} - K)^+$$

in Monte Carlo simulation. Let $X = (x_1, x_2, \ldots, x_n)^T$, Cov is the covariance matrix of X, to get random variable $X^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)})^T$, we use the Cholesky decomposition of the covariance matrix $Cov = CC^T$, so that we can generate the correlated multi-dimensional normal variable X by independent multi-dimensional normal variables Y,

$$X^{(j)} \overset{D.}{=} CY^{(j)},$$

where $Y^{(j)} = (y_1^{(j)}, y_2^{(j)}, \ldots, y_n^{(j)})^T$ is standard n-dimensional normal distribution variate with independent components, $\overset{D.}{=}$ means equal in distribution. (Cholesky decomposition is a common used method to transform independent normal distribution into correlated normal distribution, see Glasserman [30]. C is a lower triangular matrix, which makes it easier to get the conditional expectation formula in CMC method. Moreover, using programming software like matlab, we can get the Cholesky decomposition of matrix easily.)

3. Variance reduction of Monte Carlo simulation

Although Monte Carlo simulation is applicable for many problems, it has a deficiency that the convergence rate of Monte Carlo method is $O(m^{-\frac{1}{2}})$, m is the number of samples. So some variance reduction methods are needed in most problems. In this section, we proposed a hybrid variance reduction method for pricing basket options.

3.1. Splitting method

Though the closed-form formula of a basket option's price does not exist, we can get a closed-form formula of a geometric averaged basket option's price instead. A geometric averaged basket option uses the weighted geometric average instead of the arithmetic weighted average in basket option, which means the payoff is

$$(G-K)^+,$$

where $G = e^{\sum_{i=1}^{n} w_i ln(S_{iT})}$. A traditional effective way to reduce variance for pricing a basket option is using the payoff of geometric averaged option as control variate. Kemna and Vorst [7] used $(G-K)1_{\{G>K\}}$ as the control variate (CV). When CV method is used to estimate $E[X]$, the estimator is

$$X_{CV} = X - c(Y - E[Y]),$$

where Y is called a control variate, $E[Y]$ ia a known expectation of Y, c is a constant. When c equals $\frac{cov(X,Y)}{var(Y)}$, the CV method reaches the best variance reduction effect with the variance reduction factor (VRF) $\frac{1}{1-\rho_{XY}^2}$. In this paper, $(A-K)1_{\{G>K\}}$ is used as the control variate to estimate $E[(A-K)^+]$. Taking $c = 1$, then the estimator can be written as

$$(A-K)^+ - (A-K)^+1_{\{G>K\}} + E[(A-K)^+1_{\{G>K\}}] = E[(A-K)1_{\{G>K\}}] + (A-K)^+1_{\{G<K\}},$$

the equity holds because of the fact that $A \geq G$. Thus the price of basket option can be written as

$$\begin{aligned} V &= e^{-rT}(E[(A-K)1_{\{G>K\}}] + E[(A-K)^+1_{\{G<K\}}]) \\ &\equiv e^{-rT}(\xi_1 + \xi_2), \end{aligned}$$

where

$$\xi_1 = E[(A-K)1_{\{G>K\}}], \quad \xi_2 = E[(A-K)^+1_{\{G<K\}}].$$

Remark 3.1. Here we take the constant $c = 1$ instead of the optimal value $c = \frac{cov(X,Y)}{var(Y)}$ which minimize the variance of the new estimator, so that we can split the estimator into two parts. Moreover, because the control variate $(A-K)1_{\{G>K\}}$ is highly correlated with the original payoff $(A-K)^+$, $\frac{cov(X,Y)}{var(Y)}$ is close to 1 in numerical result. Taking $c = 1$ does not change the variance reduction effect of CV method a lot.

By this method, we can split the option's price into two parts. In fact, the idea of split method for pricing basket option was first proposed by Curran [26] for pricing Asian option. Curran [26] got the closed-form formula of the first part and got a lower bound of the second part to approximate the Asian option's price. Referring to Curran [26], Dingeç, Hörmann [16] regarded it as a result of control variate method. It is easy to see that the second part can be considered as a small probability event, so we propose a hybrid variance reduction method which is different from the one in [16] and especially effective for small probability events. The hybrid method we proposed in the paper combines the conditional Monte Carlo (CMC) method and the importance sampling (IS) method, since both of them seem to be effective in dealing with small probability events.

3.2. Closed form of the first part

Firstly, ξ_1 can be written as

$$\begin{aligned}\xi_1 &= E[(A-K)1_{\{G>K\}}]\\ &= E[(\sum_{i=1}^{n} w_i S_{iT} - K)1_{\{G>K\}}]\\ &= \sum_{i=1}^{n} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T} E[e^{\sigma_i\sqrt{T}x_i}1_{\{G>K\}}] - KE[1_{\{G>K\}}].\end{aligned}$$

Secondly, we get the closed-form expressions of $E[e^{\sigma_i\sqrt{T}x_i}1_{\{G>K\}}]$ and $E[1_{\{G>K\}}]$ respectively. Let

$$k = lnK - \sum_{i=1}^{n} w_i lnS_{i0} - \sum_{i=1}^{n} w_i(r - \frac{\sigma_i^2}{2} - \delta_i)T,$$

then, $G > K$ is equivalent to $\sum_{i=1}^{n} w_i\sigma_i\sqrt{T}x_i > k$. So

$$\begin{aligned}E[1_{\{G>K\}}] &= P(G>K)\\ &= P(\sum_{i=1}^{n} w_i\sigma_i\sqrt{T}x_i > k)\\ &= P(D^TX > k)\\ &= P(D^TCY > k)\\ &= \Phi(-\frac{k}{|D^TC|}),\end{aligned} \tag{3.1}$$

where $D = (w_1\sigma_1\sqrt{T}, w_2\sigma_2\sqrt{T}, \dots, w_n\sigma_n\sqrt{T})^T$, $|\cdot|$ means 2-norm for a vector, Φ is the cumulative distribution function (cdf) of the standard normal distribution. To calculate $E[e^{\sigma_i\sqrt{T}x_i}1_{\{G>K\}}]$, we denote by $C(i,:)$ the ith row of C, $Z = (z_1, z_2, \dots, z_n)^T$ is a n-dimensional independent standard normal distribution, C_2 is a $n\times n$ unit orthogonal matrix satisfying $C_2^{-1}(1,:) = \frac{D^TC}{|D^TC|}$, $C_2Z \overset{D}{=} Y$, then

$$\begin{aligned}E[e^{\sigma_i\sqrt{T}x_i}1_{\{G>K\}}] &= E[e^{\sigma_i\sqrt{T}C(i,:)Y}1_{\{D^TCY>k\}}]\\ &= E[e^{\sigma_i\sqrt{T}C(i,:)C_2Z}1_{\{D^TCC_2Z>k\}}]\\ &= E[e^{\sigma_i\sqrt{T}C(i,:)C_2Z}1_{\{z_1>\frac{k}{|D^TCC_2|}\}}].\end{aligned}$$

Let $\bar{k} = \frac{k}{|D^TCC_2|}$, $\bar{C}_i = C(i,:)C_2$, we finally get $E[e^{\sigma_i\sqrt{T}C(i,:)C_2Z}1_{\{z_1>\bar{k}\}}]$ by the following deduction,

$$\begin{aligned}E[e^{\sigma_i\sqrt{T}C(i,:)C_2Z}1_{\{z_1>\bar{k}\}}] &= \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} e^{\sigma_i\sqrt{T}\bar{C}_iZ}1_{\{z_1>\bar{k}\}}f(z_1)\dots f(z_n)dz_1\dots dz_n\\ &= \int_{\bar{k}}^{+\infty} e^{\sigma_i\sqrt{T}\bar{C}_i(1)z_1}f(z_1)dz_1\int_{-\infty}^{+\infty} e^{\sigma_i\sqrt{T}\bar{C}_i(2)z_2}f(z_2)dz_2\dots\int_{-\infty}^{+\infty} e^{\sigma_i\sqrt{T}\bar{C}_i(n)z_n}f(z_n)dz_n\\ &= e^{\frac{\sigma_i^2T\sum_{j=1}^{n}(\bar{C}_i(j))^2}{2}}\Phi(\sigma_i\sqrt{T}\bar{C}_i(1)-\bar{k}).\end{aligned} \tag{3.2}$$

The combination of above results implies the following proposition.

Proposition 3.1.

$$\xi_1 = \sum_{i=1}^{n} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T} e^{\frac{\sigma_i^2T\sum_{j=1}^{n}(\bar{C}_i(j))^2}{2}}\Phi(\sigma_i\sqrt{T}\bar{C}_i(1)-\bar{k}) - K\Phi(-\frac{k}{|D^TC|}).$$

3.3. Hybrid variance reduction method for simulation of the second part

To estimate the second part by Monte Carlo simulation, we use a hybrid variance reduction method, which combines the conditional Monte Carlo (CMC) and the importance sampling (IS) method. Numerical examples in Section 4 will show the high effectiveness of this method.

3.3.1. Conditional Monte Carlo (CMC) method

Because of the conditional variance formula

$$Var[X] = Var(E[X|Y]) + E[Var(X|Y)] \geq Var(E[X|Y]).$$

CMC method which uses conditional expectation of the estimator instead of the estimator itself can reduce variance obviously. To use the CMC method in simulation of ξ_2, we have to know the closed-form expression of the conditional expectation of ξ_2,

$$\begin{aligned}\xi_2 &= E[(A-K)^+ 1_{\{G<K\}}]\\ &= E[(A-K)1_{\{G<K<A\}}]\\ &= \sum_{i=1}^{n} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T} E[e^{\sigma_i\sqrt{T}x_i} 1_{\{G<K<A\}}] - KE[1_{\{G<K<A\}}].\end{aligned}$$

Let $\widetilde{Y} = (y_1, \ldots, y_{n-1})^T$, we try to get the conditional expectation $E[e^{\sigma_i\sqrt{T}x_i} 1_{\{G<K<A\}}|\widetilde{Y}]$ and $E[1_{\{G<K<A\}}|\widetilde{Y}]$ respectively. From last subsection, we know that $G < K$ is equivalent to $D^T CY < k$. Since C is a positive lower triangular matrix, we have

$$y_n < k_u(\widetilde{Y}) \equiv \frac{k - D^T C(:, 1:n-1)\widetilde{Y}}{D(n)C(n,n)},$$

where $C(:, 1 : n-1)$ means the first to the $(n-1)$th column of matrix C. Denote the first to the ith elements of Y by $\widetilde{Y}$. Similarly $A > K$ can be written as

$$\sum_{i=1}^{n} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{T}x_i} > K,$$

or equivalent to

$$\sum_{i=1}^{n} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{T}C(i,1:i)Y(1:i)} > K,$$

so

$$w_n S_{n0} e^{(r-\frac{\sigma_n^2}{2}-\delta_n)T+\sigma_n\sqrt{T}C(n,:)Y} > K - \sum_{i=1}^{n-1} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{T}C(i,1:i)Y(1:i)},$$

which implies that

$$y_n > k_l(\widetilde{Y}) \equiv \frac{1}{C(n,n)}\left(\frac{ln\frac{K-\sum_{i=1}^{n-1} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{T}C(i,:)Y}}{w_n S_{n0}} - (r - \frac{\sigma_n^2}{2} - \delta_n)T}{\sigma_n\sqrt{T}} - C(n, 1:n-1)\widetilde{Y}\right),$$

if $K - \sum_{i=1}^{n-1} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{T}C(i,1:i)Y(1:i)} > 0$. Taking

$$k_l(\widetilde{Y}) \equiv -\infty,$$

if $K - \sum_{i=1}^{n-1} w_i S_{i0} e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T+\sigma_i\sqrt{T}C(i,1:i)Y(1:i)} \leq 0$. So $G < K < A$ is equivalent to

$$k_l(\widetilde{Y}) < y_n < k_u(\widetilde{Y}).$$

Thus $E_{y_n}[1_{\{G<K<A\}}|\widetilde{Y}]$ can be written as

$$\begin{aligned}E_{y_n}[1_{\{k_l(\widetilde{Y})<y_n<k_u(\widetilde{Y})\}}|\widetilde{Y}] &= P(k_l(\widetilde{Y}) < y_n < k_u(\widetilde{Y}))\\ &= \Phi(k_u(\widetilde{Y})) - \Phi(k_l(\widetilde{Y})).\end{aligned}$$

By the same method, for $1 \leq i \leq n$, $E_{y_n}[e^{\sigma_i\sqrt{T}x_i} 1_{\{G<K<A\}}|\widetilde{Y}]$ can be rewritten as

$$E_{y_n}[e^{\sigma_i\sqrt{T}C(i,1:i)Y(1:i)} 1_{\{k_l(\widetilde{Y})<y_n<k_u(\widetilde{Y})\}}|\widetilde{Y}].$$

Therefore, for $i \neq n$,

$$E_{y_n}[e^{\sigma_i\sqrt{T}C(i,1:i)Y(1:i)}1_{\{k_l(\widetilde{Y})<y_n<k_u(\widetilde{Y})\}}|\widetilde{Y}] = e^{\sigma_i\sqrt{T}C(i,1:i)Y(1:i)}(\Phi(k_u(\widetilde{Y})) - \Phi(k_l(\widetilde{Y}))),$$

for $i = n$,

$$\begin{aligned} &E_{y_n}[e^{\sigma_i\sqrt{T}C(n,1:n)Y}1_{\{k_l(\widetilde{Y})<y_n<k_u(\widetilde{Y})\}}|\widetilde{Y}] \\ &= e^{\sigma_n\sqrt{T}C(n,1:n-1)\widetilde{Y}+\frac{\sigma_n^2TC(n,n)^2}{2}}(\Phi(k_u(\widetilde{Y}) - \sigma_n\sqrt{T}C(n,n)) - \Phi(k_l(\widetilde{Y}) - \sigma_n\sqrt{T}C(n,n))). \end{aligned}$$

The combination of above results leads to the following proposition.

Proposition 3.2. *Let*

$$p_i(\widetilde{Y}) = \begin{cases} e^{\sigma_i\sqrt{T}C(i,1:i)Y(1:i)}(\Phi(k_u(\widetilde{Y})) - \Phi(k_l(\widetilde{Y}))), & \text{if } i \neq n \\ e^{\sigma_n\sqrt{T}C(n,1:n-1)\widetilde{Y}+\frac{\sigma_n^2TC(n,n)^2}{2}}(\Phi(k_u(\widetilde{Y}) - \sigma_n\sqrt{T}C(n,n)) - \Phi(k_l(\widetilde{Y}) - \sigma_n\sqrt{T}C(n,n))). & \text{if } i = n. \end{cases}$$

Then we have

$$\xi_2 = E_{\widetilde{Y}}\Big[\sum_{i=1}^{n} w_iS_{i0}e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T}p_i(\widetilde{Y}) - K\Big(\Phi\big(k_u(\widetilde{Y})\big) - \Phi\big(k_l(\widetilde{Y})\big)\Big)\Big],$$

and the CMC estimator of ξ_2 *is*

$$\hat{\xi_2} = \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n}\Big(w_iS_{i0}e^{(r-\frac{\sigma_i^2}{2}-\delta_i)T}p_i(\widetilde{Y}^{(j)}) - K\Big(\Phi\big(k_u(\widetilde{Y}^{(j)})\big) - \Phi\big(k_l(\widetilde{Y}^{(j)})\big)\Big)\Big),$$

where $\widetilde{Y}^{(j)}$ *is the jth sample of simulations.*

3.3.2. Importance sampling and sample average approximation method

For further reduction of variance for simulating ξ_2, the IS technique is proposed to combined with the CMC method. Denote the conditional expectation $E_{y_n}[(A-K)^+1_{\{G<K\}}|\widetilde{Y}]$ by $h(\widetilde{Y})$ for convenience of notation. Based on the fact that

$$\begin{aligned} \xi_2 &= E[h(\widetilde{Y})] \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(\widetilde{Y})f(\widetilde{Y})d\widetilde{Y} \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(\widetilde{Y})\frac{f(\widetilde{Y})}{g(\widetilde{Y})}g(\widetilde{Y})d\widetilde{Y} \\ &= E_g[h(\widetilde{Y})\frac{f(\widetilde{Y})}{g(\widetilde{Y})}], \end{aligned}$$

where $f(\widetilde{Y})$ is the density function of $\widetilde{Y}$, $g(\widetilde{Y})$ is the new density function to be determined. IS method reduces variance by changing the measure of the random variable. The effect of variance reduction depends on the choice of the new probability density function (pdf) g. Generally, the best new pdf is chosen from a subset of all possible pdfs. In this paper, we find the best drift vector $\mu = (\mu_1, \mu_2, \ldots, \mu_{n-1})'$ to get the best new pdf, which is equivalent to

$$\underset{\mu}{argmin}\, Var_{g_\mu}[h(\widetilde{Y})\frac{f(\widetilde{Y})}{g_\mu(\widetilde{Y})}], \tag{3.3}$$

where

$$f(\widetilde{Y}) = (2\pi)^{-\frac{d}{2}}\exp(-\frac{1}{2}\|\widetilde{Y}\|^2),$$

$$g_\mu(\widetilde{Y}) = (2\pi)^{-\frac{d}{2}}\exp(-\frac{1}{2}\|\widetilde{Y}-\mu\|^2).$$

Obviously, optimization problem (3.3) is equivalent to

$$\underset{\mu}{argmin} E[h^2(\widetilde{Y})\frac{f(\widetilde{Y})}{g_\mu(\widetilde{Y})}]. \tag{3.4}$$

To solve this optimization problem, the Newton's iteration method is one of the powerful method for smooth target function. For convenience of notations, let

$$\psi(\widetilde{Y},\mu) = h^2(\widetilde{Y})\frac{f(\widetilde{Y})}{g(\widetilde{Y})},$$

$$\Psi(\mu) = E[h^2(\widetilde{Y})\frac{f(\widetilde{Y})}{g(\widetilde{Y})}],$$

then optimization problem (3.4) can be rewritten as

$$\underset{\mu}{argmin}\, \Psi(\mu). \tag{3.5}$$

Assumption 3.1. For any $\mu \in \mathbb{R}^d, d \geq 1, p > 1, E_f[h^{2p}(\widetilde{Y})e^{-p\mu\cdot\widetilde{Y}}] < \infty$ and $E[\|\widetilde{Y}\|^{2p}h^{2p}(\widetilde{Y})e^{-p\mu\cdot\widetilde{Y}}] < \infty$ always hold.

Theorem 3.1. *Under Assumption 3.1, we have*

$$\nabla_\mu \Psi(\mu) = E[\nabla_\mu \psi(\widetilde{Y}, \mu)],$$

$$Hess_\mu[\Psi(\widetilde{Y}, \mu)] = E\big[Hess_\mu[\psi(\widetilde{Y}, \mu)]\big],$$

where $\nabla_\mu \Psi(\mu)$ and $Hess_\mu[\Psi(\widetilde{Y}, \mu)]$ are the gradient and Hessian matrix of Ψ respectively, $\nabla_\mu \psi(\widetilde{Y}, \mu)$ and $Hess_\mu[\psi(\widetilde{Y}, \mu)]$ are the gradient and Hessian matrix of ψ respectively.

Proof. From Lemma 2 in [17], we know that under Assumption 3.1, we have

$$\sup_{\mu\in\mathbb{R}^d} E[\big((\mu - \widetilde{Y})h^2(\widetilde{Y})\exp(-\mu\widetilde{Y} + \frac{1}{2}\|\mu\|^2)\big)^p] < \infty,$$
$$\sup_{\mu\in\mathbb{R}^d} E[\big((I_d + (\mu - \widetilde{Y})(\mu - \widetilde{Y})')h^2(\widetilde{Y})\exp(-\mu Y + \frac{1}{2}\|\mu\|^2)\big)^p] < \infty.$$

Therefore, $\nabla_\mu \psi(\widetilde{Y}, \mu)$ and $Hess_\mu[\psi(\widetilde{Y}, \mu)]$ are all uniformly integrable. So

$$\nabla_\mu \Psi(\mu) = E[\nabla_\mu \psi(\widetilde{Y}, \mu)],$$

$$Hess_\mu[\Psi(\widetilde{Y}, \mu)] = E[Hess_\mu[\psi(\widetilde{Y}, \mu)]]. \quad \square$$

The existence and uniqueness of solution of problem (3.4) is based on the following Theorem.

Theorem 3.2. *Under Assumption 3.1, the function $\Psi(\mu)$ is C^∞ in $\mathbb{R}^d$ and there exists a unique $\mu^* \in \mathbb{R}^d$ such that*

$$\Psi(\mu^*) = \min_{\mu\in\mathbb{R}^d} \Phi(\mu).$$

Proof. By calculating directly, we have

$$\nabla_\mu \Psi(\mu) = E[(\mu - \widetilde{Y})h^2(\widetilde{Y})\exp(-\mu\cdot\widetilde{Y} + \frac{1}{2}\|\mu\|^2)],$$

$$Hess_\mu[\Psi(\mu)] = E[(I_d + (\mu - \widetilde{Y})\cdot(\mu - \widetilde{Y})')h^2(\widetilde{Y})\exp(-\mu\cdot\widetilde{Y} + \frac{1}{2}\|\mu\|^2)].$$

Obviously we can obtain derivatives of any orders. To solve optimization problem (3.5), we set $\nabla_\mu \Psi(\mu) = \vec{0}$ and obtain a system of nonlinear equations as following

$$\frac{\partial \Phi(\mu)}{\partial \mu_i} = E[(\mu_i - y_i)h^2(\widetilde{Y})e^{-\mu\cdot\widetilde{Y}+\frac{1}{2}\|\mu\|^2}] = 0, \quad (i = 1, 2, \ldots, d). \tag{3.6}$$

Let $a_{ij} = \frac{\partial^2\Phi(\mu)}{\partial\mu_i\partial\mu_j}$, then we have

$$a_{ij} = (\frac{1}{\sqrt{2\pi}})^d \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (\delta_{ij} + (\mu_i - y_i)(\mu_j - y_j))h^2(\widetilde{Y})\exp(-\|\widetilde{Y}\|^2 + \frac{1}{2}\|\widetilde{Y} - \mu\|^2)d\widetilde{Y},$$

where δ_{ij} is the Kronecker function. So for any $z \in \mathbb{R}^d$, it can be easily checked that

$$\begin{aligned}\sum_{i,j} a_{ij}z_iz_j &= (\frac{1}{\sqrt{2\pi}})^d \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (\sum_{i=1}^d z_i^2 + (\sum_{i=1}^d (\mu_i - y_i)z_i)^2)h^2(\widetilde{Y})\cdot\exp(-\|\widetilde{Y}\|^2 + \frac{1}{2}\|\widetilde{Y} - \mu\|^2)d\widetilde{Y} \\ &\geq (\frac{1}{\sqrt{2\pi}})^d \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h^2(\widetilde{Y})\exp(-\|\widetilde{Y}\|^2)d\widetilde{Y} \sum_{i=1}^d z_i^2.\end{aligned}$$

Therefore, $A = (a_{ij})$ is a positive definite matrix and the minimum eigenvalue of the matrix is no less than some positive constant that is irrelevant to μ, which implies that

$$\|A^{-1}\| \le C,$$

where constant C is also independent with μ. Consequently, according to Theorem 7.1 in [31], optimization problem (3.5) is equivalent to nonlinear system (3.6), and we can use Newton's method to solve this nonlinear system. Moreover the solution of nonlinear system (3.6) is unique. Thus we complete the proof of Theorem 3.2. □

Theorem 3.2 shows the existence and uniqueness of the optimization problem (3.5), it also tells us that the Newton's iteration scheme

$$\mu^{(i+1)} = \mu^{(i)} - Hess_{\mu}^{-1}[\Psi(\mu^{(i)})]\nabla_{\mu}\Psi(\mu^{(i)}) \tag{3.7}$$

is a feasible method to solve this problem. Because of Theorem 3.1, we can use sample average approximation (SAA) method to use the average value of samples instead of the expectation value while solving the optimization problem in practice, which means that we should solve the following optimization problem

$$\underset{\mu\in\mathbb{R}^d}{argmin}\ \frac{1}{m}\sum_{j=1}^{m}\Psi(\widetilde{Y}^{(j)}, \mu) \tag{3.8}$$

instead of solving the optimization problem (3.5). Next, we will prove the convergence and convergence rate of the SAA method by using some theorems in [32].

Theorem 3.3. *Let μ^* and μ_m^* be the optimal solution of optimization problem (3.5) and (3.8) respectively, v^* and v_m^* are optimal values of them. Assume $\psi(Y, \mu)$ and $\nabla_{\mu}\psi(Y, \mu)$ are Lipschitz continuous with respect to μ. Then we have*

$$\|\mu^* - \mu_m^*\| = O(m^{-\frac{1}{2}}),$$

$$\|v^* - v_m^*\| = O(m^{-\frac{1}{2}}).$$

Proof. From Assumption 3.1, we know that $E[\psi^2(Y, \mu)] < \infty$, so by the Theorem 11 in Kim, Pasupathy and Henderson [32], we have

$$\|v^* - v_m^*\| = O(m^{-\frac{1}{2}}).$$

Also from Assumption 3.1, we know that $E[\|\nabla_{\mu}\psi(Y, \mu)\|^2] < \infty$. By the Theorem 12 in Kim, Pasupathy and Henderson [32], we have

$$\|\mu^* - \mu_m^*\| = O(m^{-\frac{1}{2}}). \quad \square$$

Remark 3.2. We will discuss the rationality about the assumptions in the above theorems. Because function $h(\widetilde{Y})$ is a truncated exponential function, $\widetilde{Y}$ is assumed to be a normal distribution in this paper. As we know, exponential function of a normal distribution variable is integrable, so Assumption 3.1 is satisfied. In the proof of Theorem 3.3, we use the Lipschitz property of $\psi(Y, \mu)$ and $\nabla_{\mu}\psi(Y, \mu)$ with respect to μ. It is reasonable because the $-\mu^2 + c\mu$ is only on the exponent of $\psi(Y, \mu)$, where c is a constant. It is easy to check the Lipschitz property is also valid for functions $\psi(Y, \mu)$ and $\nabla_{\mu}\psi(Y, \mu)$.

3.3.3. A prediction–correction iteration method

Although the Newton's method is applicable for many nonlinear systems. However, it may take large computation costs for high-dimensional problems, since we have to compute $Hess_{\mu}^{-1}[\Psi(\mu^{(i)})]$ at every iteration step. We proposed a prediction–correction (PC) method to get the best drift vector efficiently. It is well known that if the new pdf satisfies $g_{\mu}(\widetilde{Y}) = c\, h(\widetilde{Y})f(\widetilde{Y})$, for some constant c, then IS estimation has zero variance. Although this pdf is hard to get, we know that if the shape of $g_{\mu}(\widetilde{Y})$ is similar with $h(\widetilde{Y})f(\widetilde{Y})$, the variance reduction effect will be good. Our PC method consists of two steps. The first step is to use moment match method to get the prediction value of the optimal drift vector μ by solving the equations

$$\frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} y_i^k h(\widetilde{Y})f(\widetilde{Y})dy_1\ldots dy_{n-1}}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} y_i^k g_{\mu}(\widetilde{Y})dy_1\ldots dy_{n-1}} = c,\ \ i = 1,\ldots,n-1,\ \ k = 0, 1. \tag{3.9}$$

Moment estimation is a general method in statistics. We know from Theorem 7 in section 12 of chapter II, Shiryaev [33] that two distributions are exactly same if all order of moments of them are equivalent. Solving equations (3.9), for $i = 1, 2, \ldots, n-1$, we get

$$\mu_i = \frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} y_i h(\widetilde{Y})f(\widetilde{Y})dy_1\ldots dy_{n-1}}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h(\widetilde{Y})f(\widetilde{Y})dy_1\ldots dy_{n-1}}. \tag{3.10}$$

Algorithm 1 Hybrid variance reduction method

Input: The parameters of the option and assets: $K, T, w, S_0, \delta, \sigma, n, m$.
Output: Option price V.

1: Calculate the expectation of the first part ξ_1 using the method in section 2.2;
2: Calculate the conditional expectation of the second part $h(\widetilde{Y})$ using the method in section 2.3.1;
3: Calculate the prediction value of μ by solving equations 3.9
4: Calculate the correction value of μ by iteration scheme 3.11
5: Calculate the IS estimation of ξ_2, $\xi_2 \approx \frac{1}{m}\sum_{j=1}^{m} h(\widetilde{Y}^{(j)})\frac{f(\widetilde{Y}^{(j)})}{g_\mu(\widetilde{Y}^{(j)})}$;
6: **return** $V = e^{-rT}(\xi_1 + \xi_2)$.

The integrals in Eq. (3.10) can be estimated by Monte Carlo simulations. Because we have already generated samples before, this step takes only a few computation time. The second step is the correction process. We use the value of μ_i getting from the first step as the initial value, use the iteration scheme

$$\mu_i^{(j+1)} = \mu_i^{(j)} - (\frac{\partial^2 \Psi}{\partial \mu_i^2}(\mu_i^{(0)}))^{-1}\frac{\partial \Psi}{\partial \mu_i}(\mu_i^{(j)}) \tag{3.11}$$

to get the correction value of μ_i. Because this iteration scheme only need to compute the inverse matrix one time, so it can save a lot computation cost for high dimensional problems. In iteration scheme (3.11), SAA method is also used to estimate the gradient and Hessian matrix respectively. The convergence of the iteration process can be obtained by the Theorem 6.1 of Quarteroni, Sacco and Saleri [31].

Theorem 3.4 (*Theorem 6.1 of Quarteroni, Sacco and Saleri [31]*). *Consider the iteration sequence $\mu^{(k+1)} = \phi(\mu^{(k)})(k = 0, 1, \ldots)$. Assume that*

1. $\phi : \mathbb{R}^d \to \mathbb{R}^d$;
2. $\phi \in C^1(\mathbb{R}^d)$;
3. $\exists 0 < K < 1 : \|\phi'(\mu)\| < K, \ \forall \mu \in \mathbb{R}^d$.

Then, ϕ has a unique fixed point μ^* in $\mathbb{R}^d$ and the sequence $\{\mu^{(k)}\}$ converges to μ^* for any choice of $\mu^{(0)} \in \mathbb{R}$. Moreover, we have

$$\lim_{k\to\infty} \frac{\|\mu^{(k+1)} - \mu^*\|}{\|\mu^{(k)} - \mu^*\|} = \|\phi'(\mu^*)\|.$$

In above prediction–correction iteration scheme (3.11),

$$\phi'(\mu) = I - \left(Hess_\mu[\Psi(\mu^{(0)})]\right)^{-1} Hess_\mu[\Psi(\mu)],$$

where I is a unit matrix. Since the function $\Psi(\mu)$ is convex in $\mathbb{R}^d$, so $\Psi(\mu)$ exists a unique minimal point in $\mathbb{R}^d$. Moreover, thanks to the fact that the initial value $\mu^{(0)}$ in our PC iteration algorithm is close to the optimal value μ^*, i.e. $\mu^{(0)} \approx \mu^*$, thus ϕ in PC algorithm satisfies the assumption of Theorem 3.4, which means the iteration in our PC algorithm has linear convergence with $\|\phi'(\mu^*)\| = \|I - (Hess_\mu[\Psi(\mu^{(0)})])^{-1} Hess_\mu[\Psi(\mu^*)]\|$.

(By using PC algorithm, we can get the optimal parameters of IS method using less computation costs without decline of accuracy. The PC algorithm can be seen as a two-step method. The first step is prediction by moment estimation method. The second step is correction by the iteration scheme (3.11). The moment estimation method is a traditional method in statistics and we first use this method in IS method. In numerical results, we find even the parameters by only moment estimation method can achieve satisfying variance reduction effect. Meanwhile, the moment estimation method takes much less computation costs. So only using the moment estimation is also efficiency in practice. In addition of moment estimation, we add the correction step to get more accuracy parameters. The variance reduction effect by parameters after correction step can be same with Newton's iteration method. What is more, although this method is under GBM model, this method can also be used in other assets models like Levy process, stochastic rate models, stochastic volatility models, etc.)

Remark. In this paper, we add a drift vector to the standard normal distribution which is equivalent to change the mean vector of the standard normal distribution in IS method. In fact, we can also change the variance vector of the standard normal distribution in IS method by using the method we proposed. However, the variance reduction effect does not change a lot with or without changing variance in this case. We will show this fact numerically in the next section. While extra computation cost is need to calculate variance vector. So we just add drift vector to standard normal distribution here.

4. Numerical results

4.1. Seven-dimensional basket option

In order to compare our method with Milevsky and Posner [27] and Dingeç and Hörmann [16], we use the same G-7 index-linked guaranteed investment certificates (ILGICs) as an example. For more details about ILGICs see [27]. The parameters in

Table 1
Parameters of the assets.

i	w_i	σ_i	δ_i	ρ_{i1}	ρ_{i2}	ρ_{i3}	ρ_{i4}	ρ_{i5}	ρ_{i6}	ρ_{i7}
1	0.10	0.1155	0.0169	1.00	0.35	0.10	0.27	0.04	0.17	0.71
2	0.15	0.2068	0.0239	0.35	1.00	0.39	0.27	0.50	−0.08	0.15
3	0.15	0.1453	0.0136	0.10	0.39	1.00	0.53	0.70	−0.23	0.09
4	0.05	0.1799	0.0192	0.27	0.27	0.53	1.00	0.46	−0.22	0.32
5	0.20	0.1559	0.0081	0.04	0.50	0.70	0.46	1.00	−0.29	0.13
6	0.10	0.1462	0.0362	0.17	−0.08	−0.23	−0.22	−0.29	1.00	−0.03
7	0.25	0.1568	0.0166	0.71	0.15	0.09	0.32	0.13	−0.03	1.00

Table 2
Numerical results 1.

T	K	V_{MC}	V_{IS}	VRF_{IS}	V_{NE}	VRF_{NE}
1	80	23.1831	23.1103	19.5905	23.1412	2.0464×10^7
	100	6.2030	6.2643	7.9931	6.2217	2.7323×10^4
	120	0.3373	0.3491	33.5433	0.3536	6.6096×10^3
2	80	25.9139	26.0286	13.8856	26.0424	1.3817×10^6
	100	10.1512	10.1769	7.8114	10.2172	9.9123×10^3
	120	2.1051	2.0478	14.8567	2.0543	2.6072×10^3
3	80	28.5589	28.7405	14.7744	28.6994	2.5082×10^5
	100	13.5486	13.6777	8.3799	13.7437	6.7683×10^3
	120	4.4638	4.4528	10.5266	4.4532	1.5429×10^3

Table 3
Numerical results 2.

T	K	V_{ME}	VRF_{ME}	V_{PC}	VRF_{PC}	$V_{\mu,\sigma}$	$VRF_{\mu,\sigma}$
1	80	23.1412	3.1349×10^7	23.1412	3.1349×10^7	23.1412	4.2400×10^7
	100	6.2222	2.3848×10^4	6.2222	2.4119×10^4	6.2218	5.2782×10^4
	120	0.3535	6.7585×10^3	0.3536	6.6252×10^3	0.35372	9.8004×10^4
2	80	26.0424	1.2634×10^6	26.0424	1.2634×10^6	26.0425	1.8125×10^6
	100	10.2170	9.3182×10^3	10.2172	9.9308×10^3	10.2149	2.0112×10^4
	120	2.0537	2.4484×10^3	2.0543	2.6105×10^3	2.0546	4.0913×10^3
3	80	28.6996	2.1594×10^5	28.6996	2.1594×10^5	28.6990	4.9969×10^5
	100	13.7430	6.2499×10^3	13.7436	6.8005×10^3	13.7433	1.2160×10^4
	120	4.4535	1.5073×10^3	4.4533	1.5346×10^3	4.4533	9.8004×10^3

the example are provided by the J.P. Morgan Risk Metrics system on July 17, 1997. Taking interest rate $r = 0.063$, initial price $S_i(0) = 100$, $i = 1, \dots, 7$ as in the previous papers. Sample size is $m = 10{,}000$. Other parameters about the assets are presented in Table 1.

We list the numerical results in Tables 2 and 3 for various strike prices K and maturities T, V_{MC} is the option price by traditional Monte Carlo method, V_{IS} is the option prices by usual importance sampling method with drift measure change to the payoff $(\sum_{i=1}^{n} w_i S_{iT} - K)^+$. V_{NE}, V_{ME}, V_{PC}, $V_{\mu,\sigma}$ are the option prices by hybrid variance reduction method, the different indexes represent the prices by different methods to solve the corresponding optimization problem in IS, V_{NE} is by the Newton's method, V_{ME} is by the moment estimation method, V_{PC} is by the PC method, $V_{\mu,\sigma}$ is by both changing the mean value and the variance vector. We use VRF(variance reduction factor) to judge the variance reduction effect, i.e.

$$VRF_{IS} = \frac{Var_{MC}}{Var_{IS}},\ VRF_{NE} = \frac{Var_{MC}}{Var_{NE}},\ VRF_{ME} = \frac{Var_{MC}}{Var_{ME}},$$

$$VRF_{PC} = \frac{Var_{MC}}{Var_{PC}}, VRF_{\mu,\sigma} = \frac{Var_{MC}}{Var_{\mu,\sigma}}.$$

It is clear that the hybrid variance reduction method has much better variance reduction effect than direct IS method to simulate ξ_2, since the VRF of hybrid variance reduction method is about $10^3 - 10^7$ for different strikes and maturities, while for direct IS method it is less than 50. On the other hand, the VRF values are almost the same for IS method by different methods to solve the corresponding optimization problem. However the moment estimation and PC method we propose in this paper take much less computation time than the Newton's method. If we use IS method to simulate ξ_2 by changing both the mean value and the variance vector of the normal distribution, the VRF is a little better than by only changing the mean value, although it takes much more computation cost.

(In order to show the effectiveness of PC method, Fig. 1 shows the parameter of IS method by moment estimation, Newton's iteration method and PC algorithm separately under $K = 100$, $T = 1$. As the fact that Newton's iteration method is effect for this problem, we can regard the parameters by Newton's iteration method as real value of the parameters. It is

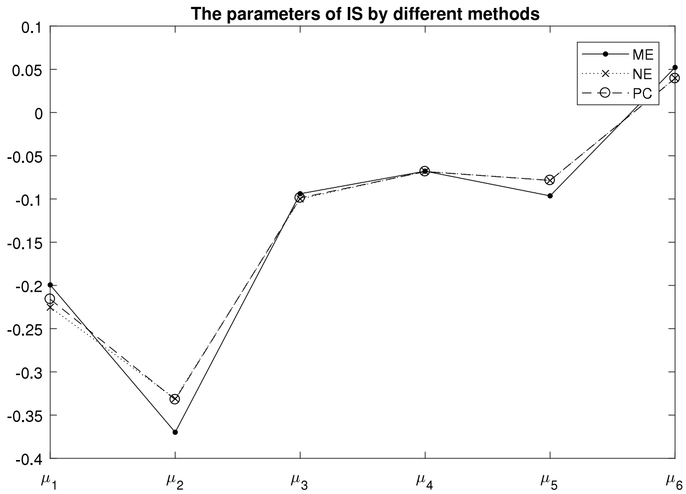

Fig. 1. The relationship between log(VRF) and volatility $s = \sigma_1 = \sigma_2$.

Table 4
Parameters of the assets.

i	w_i	σ_i	δ_i	ρ_{i1}	ρ_{i2}
1	0.5	s	0	1.00	ρ
2	0.5	s	0	ρ	1.00

clear that the parameters by moment estimation are close to real value and the parameters by PC algorithm are almost same with real value.)

Remark. Compared with the method in [16], we use an IS variance reduction technique based on the conditional expectation to simulate ξ_2, avoiding to solve many nonlinear equations in simulations. Dingeç and Hörmann [16] used control variate technique based on a conditional expectation, where the Newton's method was used for every sample.

4.2. The influence of some parameters to the variance reduction effect

In this subsection, we try to test numerically the sensitivities of variance reduction effect with respect to some parameters of the option. In fact, numerical results in Section 4.1 show that the variance reduction effect is influenced by the value of strike price and maturity of the option. The volatility and the correlation coefficient of the assets are also very important parameters besides strike price and maturity. The values of parameters for a two-dimensional basket option are given in Table 4.

Let $\sigma_1 = \sigma_2 = s$, we first keep $\rho = 0.2$ fixed and change the value of s from 0.01 to 0.99. The values of VRF for various values of s are plotted in Fig. 2.

It is clear that the variance reduction effect becomes better as the volatility of the assets becomes smaller. Although the VRF seems small when s is close to 1 in Fig. 2, the value of VRF is still over 1000. So this method is effective to assets with various volatilities in practice.

Then we keep $s = 0.2$ fixed, and change the value of ρ from -0.99 to 0.99. The numerical results are shown in Fig. 3.

Similarly, we can see from Fig. 3 that the variance reduction effect becomes better as the correlation coefficient ρ becomes larger, and our method is also satisfied.

5. Conclusion

In this paper, we proposed a hybrid variance reduction method for pricing basket options by splitting the estimator of their payoff into two parts: the first part is the closed-form solution, and the second part is the expectation of a rare event. The conditional Monte Carlo method, importance sampling Monte Carlo method are then used for simulating the second part. Those two methods have good effect on reducing the variance for solving the expectation of rare events. The conditional Monte Carlo method can be applied benefit from the estimator of the first part can be written as a conditional expectation of a variable. Theoretically, the conditional Monte Carlo method can reduce the variance of simulation and save the costing time.

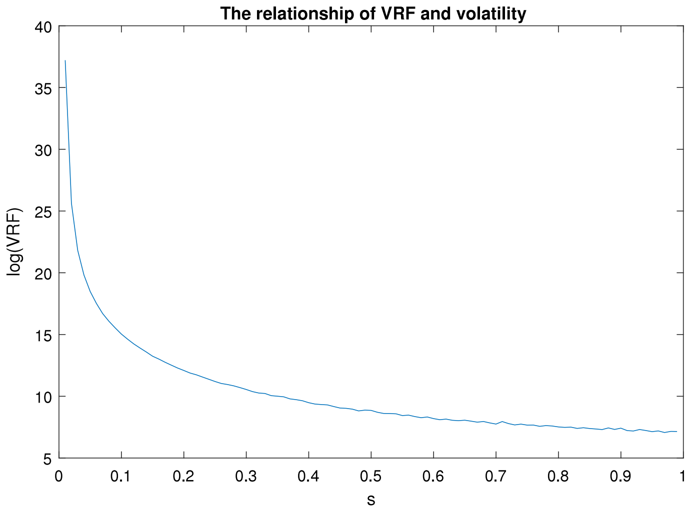

Fig. 2. The relationship between log(VRF) and volatility $s = \sigma_1 = \sigma_2$.

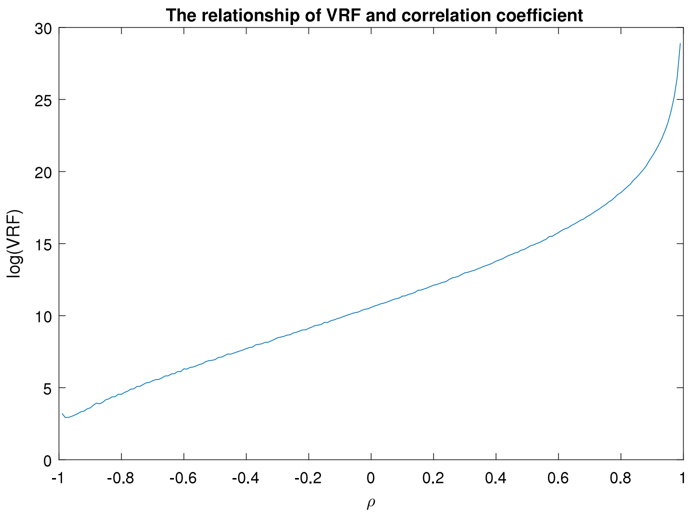

Fig. 3. The relationship between log(VRF) and correlation coefficient ρ.

When solving the optimal parameters of the importance sampling Monte Carlo method based on the drift measure changing, we adopt a new prediction–correction (PC) iterative method based on the idea of moment matching, which cannot only maintain the convergence properties of the optimization problem, but also can greatly save the computing time. Numerical results show that our method has great variance reduction effect.

It can be seen from the paper that the conditional expectation formula transforms the piecewise smooth function into a smooth function with any order of derivative, which can be used to take derivatives to some parameters of estimator directly. Therefore, it is useful to analyze the sensitivity of the parameters. Our method has a very good variance reduction effect, mainly due to the characteristics of the payoff function of basket options, so the same method is also applicable to the pricing of like Asian options and discrete sampling Lookback options. (How to extend conditional Monte Carlo method to other derivatives like barrier options. And how to use this method for assets under other models like jump diffusion model is also an interesting question. We would also like to extend this method to other areas like management and civil engineering.)

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References

[1] J. Topper, Finite element modeling of exotic options, in: Operations Research Proceedings 1999, Springer, Berlin Heidelberg, 2000, pp. 336–341.
[2] P. Lötstedt, J. Persson, L. Von Sydow, et al., Space–time adaptive finite difference method for European multi-asset options, Comput. Math. Appl. 53 (8) (2007) 1159–1180.
[3] S. Borovkova, F.J. Permana, H.V.D. Weide, A closed form approach to the valuation and hedging of basket and spread option, J. Deriv. 14 (4) (2007) 8–24.
[4] C.C.W. Leentvaar, C.W. Oosterlee, Multi-asset option pricing using a parallel Fourier-based technique, J. Comput. Finance 12 (1) (2008) 1–26.
[5] E. Larsson, K. Ahlander, A. Hall, Multi-dimensional option pricing using radial basis functions and the generalized Fourier transform, Journal of Computational & Applied Mathematics 222 (1) (2008) 175–192.
[6] T. Jiang, X. Liu, Z. Yu, Finite element algorithms for pricing 2-D basket options, in: 2009 1st International Conference on Information Science and Engineering (ICISE), IEEE, 2009, pp. 4881–4886.
[7] A.G.Z. Kemna, A.C.F. Vorst, A pricing method for options based on average asset values, J. Banking Finance 14 (1) (1990) 113–129.
[8] P. Pellizzari, Efficient Monte Carlo pricing of basket options, Finance (1998).
[9] M.B. Giles, Multilevel Monte Carlo for basket options, in: Winter Simulation Conference, IEEE, 2009, pp. 1283–1290.
[10] P. Glasserman, P. Heidelberger, P. Shahabuddin, Asymptotically optimal importance sampling and stratification for pricing path-dependent options, Math. Finance 9 (2) (1999) 117–152.
[11] C. Joy, P.P. Boyle, K.S. Tan, Quasi-Monte Carlo methods in numerical finance, Manage. Sci. 42 (6) (1996) 926–938.
[12] N.C. Petroni, P. Sabino, Pricing and hedging asian basket options with quasi-Monte Carlo simulations, Methodol. Comput. Appl. Probab. 15 (1) (2013) 147–163.
[13] Y.H. Hu, D.Q. Chen, Parallel randomized quasi-Monte Carlo simulation for asian basket option pricing, J. Algorithms Comput. Technol. 6 (1) (2012) 101–112.
[14] S. Kim, R. Pasupathy, S.G. Henderson, A guide to sample average approximation, in: Handbook of Simulation Optimization, Springer, New York, 2015, pp. 207–243.
[15] R. Korn, S. Zeytun, Efficient basket Monte Carlo option pricing via a simple analytical approximation, J. Comput. Appl. Math. 243 (1) (2013) 48–59.
[16] K.D. Dingeç, W. Hörmann, Control variates and conditional Monte Carlo for basket and Asian options, Insurance Math. Econom. 52 (3) (2013) 421–434.
[17] G.X. Jiang, C.L. Xu, M.C. Fu, On sample average approximation algorithms for determining the optimal importance sampling parameters in pricing financial derivatives on Levy processes, Oper. Res. Lett. 44 (1) (2016) 44–49.
[18] S. Zhang, K.W. Chau, Dimension reduction using semi-supervised locally linear embedding for plant leaf classification, in: International Conference on Intelligent Computing, Springer, Berlin, Heidelberg, 2009, pp. 948–955.
[19] K.W. Chau, C.L. Wu, A hybrid model coupled with singular spectrum analysis for daily rainfall prediction, J. Hydroinform. 12 (4) (2010) 458–473.
[20] R. Taormina, K.W. Chau, Data-driven input variable selection for rainfallCrunoff modeling using binary-coded particle swarm optimization and Extreme Learning Machines, J. Hydrol. 529 (2015) 1617–1632.
[21] L. Hong, A.S. Doumith, B.D. Davison, Co-factorization machines:modeling user interests and predicting individual decisions in Twitter, in: ACM International Conference on Web Search and Data Mining, ACM, 2013, pp. 557–566.
[22] R. Rackwitz, A. Joanni, Risk acceptance and maintenance optimization of aging civil engineering infrastructures, Struct. Saf. 31 (3) (2009) 251–259.
[23] W.C. Wang, K.W. Chau, D.M. Xu, et al., Improving forecasting accuracy of annual runoff time series using ARIMA based on EEMD decomposition, Water Resour. Manage. 29 (8) (2015) 2655–2675.
[24] J. Zhang, K.W. Chau, Multilayer ensemble pruning via novel multi-sub-swarm particle swarm optimization, Comput. Inf. Sci. 1 (2) (2009) 1362–1381.
[25] X.Y. Chen, K.W. Chau, A.O. Busari, A comparative study of population-based optimization algorithms for downstream river flow forecasting by a hybrid neural network model, Eng. Appl. Artif. Intell. 46 (2015) 258–268.
[26] M. Curran, Valuing asian and portfolio options by conditioning on the geometric mean price, Manage. Sci. 40 (12) (1994) 1705–1711.
[27] M.A. Milevsky, S.E. Posner, A closed-form approximation for valuing basket options, J. Deriv. 5 (4) (1998) 54–61.
[28] A. Leccadito, T. Paletta, R. Tunaru, Pricing and hedging basket options with exact moment matching, Insurance Math. Econom. 69 (2) (2016) 59–69.
[29] R. Caldana, G. Fusai, A. Gnoatto, et al., General closed-form basket option pricing bounds, Quant. Finance 16 (4) (2016) 535–554.
[30] P. Glasserman, Monte Carlo Methods in Financial Engineering, Springer Science & Business Media, 2004.
[31] A. Quarteroni, R. Sacco, F. Saleri, Numerical Mathematics, Springer Science & Business Media, 2010.
[32] S. Kim, R. Pasupathy, S.G. Henderson, A guide to sample average approximation, in: Handbook of Simulation Optimization, Springer, New York, 2015, pp. 207–243.
[33] A.N. Shiryaev, Probability, second ed., Springer, Berlin, 1996.