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Least-square-based control variate method for pricing options under general factor models

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ABSTRACT

This paper proposes a class of simple but efficient control variate method for pricing derivatives under multiple factor models including stochastic volatility and stochastic interest rate model. The control variate can help us to obviously reduce the error of Monte Carlo simulation. Briefly speaking, we construct a virtual asset with deterministic volatility and deterministic interest rate which has high correlation with the original underlying asset based on the method of least square, and use derivative written on the virtual asset as control variate in pricing derivative written on the original underlying asset. Some theoretic results can help us to understand the mechanism of a control variate. Numerical examples show that simulation error is significantly reduced by our method. The advantage of our method is that it has no analytic form request for the underlying asset model, so the method is flexible to deal with and broadly applicable for derivative pricing.

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1. Introduction

With the fast development of financial market, the assumption of deterministic volatility and deterministic interest rate of underlying asset is unacceptable in many situations [1]. In some situations, such as a long term or rate-sensitive financial derivative, both interest rate and volatility should be assumed stochastic. There are many multi-factor stochastic models to match different needs in financial market, such as the Hull–White [15] stochastic volatility model and the Heston [14] stochastic volatility model, the extension of the Heston model with stochastic interest rate and stochastic volatility, see, e.g. Grzelak and Oosterlee [12] and Grzelak *et al.* [13].

In most situations, the underlying asset or the payoff of the derivative is complicated, so there is hardly an analytic price formula. Therefore, many researching works have been done for pricing financial derivatives by some numerical methods, such as the binomial tree method, Fourier transform method and methods based on partial differential equation [6]. However, all the methods mentioned above will encounter essential difficulties for the higher dimension cases. In this case, a Monte Carlo method for pricing corresponding financial derivative is adapted. However, Monte Carlo simulation usually means large amount of computing and takes a longer time. So several variance reduction techniques have been proposed to improve the efficiency of Monte Carlo simulation, such as control variate, important sampling, antithetic variate, a moment matching method and conditional Monte Carlo method. A control variate method is an effective and widely used method in financial applications.

The earliest application of the control variate technique to option pricing is due to Boyle [2]. Kemna and Vorst [16], Broadie and Glasserman [3], Chiarella *et al.* [5] used control variate in option pricing. An appropriate control variate is the key in the control variate method. Kemna and Vorst [16], Dingec *et al.* [7] used geometric average Asian option as control variate in pricing arithmetic average Asian option. Pellizzari [18] used a European option of single asset as control variate in pricing multivariate option. However, if the underlying asset has stochastic volatility or/and stochastic interest rate, even a simple derivative such as a European option or geometric average option has no closed-form solution (although semi-closed-form pricing expressions had been established by Hull and White [15], Heston [14], respectively), so it cannot be used as a control variate. Some authors, such as Glasserman [11], suggested to construct a control variate written as a virtual asset with constant volatility. Ma and Xu [17] improved the result by constructing an efficient control variate in a stochastic volatility model, but their work dependent on a specific form of the volatility model, it may not easily extend to more complicated multi-factor models. Fouque and Han [8] proposed a two-step strategy to reduce the variance for pricing options. Chiarella *et al.* [5] and Fouque and Han [9] constructed another kind of powerful control variate framework. For pricing basket options, an accurate lower bound based on an approximating set and a fast bounded approximation based on the arithmetic-geometric mean inequality was constructed for a general class of continuous-time financial models [4,10], which can also be used as an efficient control variate to improve Monte Carlo simulation accuracy.

In this paper, we will construct a virtual asset with deterministic volatility and interest rate based on the least-square method, which has high correlation with the original underlying asset in the whole path. This feature makes the derivatives written on the virtual asset to be used as a control variate in pricing derivatives written on the original underlying asset. In practical applications, we can use the virtual asset itself, its corresponding European option, geometric average option written on the virtual asset as the control variates.

The rest of the paper is organized as follows. In the next section, we will construct the control variates under three typical asset models, i.e. the stochastic volatility model, stochastic interest rate model, and the hybrid model with stochastic interest rate and stochastic volatility simultaneously, some key formulas are established. Numerical tests are given in Section 3 for several typical options such as an Asian option with arithmetic average value of asset at the discrete-sampled observing points, European option under the multiple stochastic factors asset model, which show the high efficiency of our method. Section 4 concludes the methods in the paper. The method in this paper may also applicable to some more complicated exotic options in financial practice.

2. Construction of control variate

2.1. Basic principle of control variate method

We will do some brief introduction of how the control variate method works at variance reduction, for more details, see, e.g. Glasserman [11].

Let V be a random variable, the classical Monte Carlo estimator of $E[V]$ is

$$\bar{V} = \frac{1}{m} \sum_{j=1}^m V_j,$$

where the V_j are i.i.d. copies of V . The random variable \bar{V} has the same expectation as V and variance $(1/m) \text{Var}[V]$. X is called a control variate of V if (i) $E[X]$ is known, (ii) X is correlated with V . Let $b \in \mathbb{R}$, then random variable

$$V(b) = V - b(X - E[X])$$

has expectation $E[V]$, and its variance is minimized at

$$b^* = \frac{\text{Cov}[V, X]}{\text{Var}[X]}$$

with

$$\text{Var}[V(b^*)] = (1 - \rho_{XV}^2) \text{Var}[V], \quad (1)$$

where ρ_{XV} is the correlation coefficient between X and V . So we have another estimator of $E[V]$

$$\bar{V}(b^*) = \frac{1}{m} \sum_{j=1}^m V_j(b^*).$$

where $\bar{V}(b^*)$ also has the same expectation as V , however, its variance is replaced by $(1 - \rho_{XV}^2) \text{Var}[\bar{V}]$. It is obvious that $\bar{V}(b^*)$ is a better estimator than \bar{V} . Since b^* is unknown, we generally use its estimator

$$\hat{b} = \frac{\sum_{j=1}^m (V_j - \bar{V})(X_j - \bar{X})}{\sum_{j=1}^m (X_j - \bar{X})^2}$$

to replace it.

Unfortunately, there is no uniform method to construct the control variate, we now discuss it in two special cases. Let V_T be the payoff of an option, X is the corresponding control variate for simulating $E[V_T]$ and $V(b) = V_T - b(X - E[X])$. It is easy to check that

$$\text{Var}[V(b^*)] = \min_b \text{Var}[V(b)] \leq \text{Var}[V(1)] = \text{Var}[V_T - X] \leq E[(V_T - X)^2]. \quad (2)$$

The combination of Equations (1) and (2) leads to that

$$0 \leq 1 - \rho_{XV}^2 \leq \frac{E[(V_T - X)^2]}{\text{Var}[V_T]}. \quad (3)$$

Case (i). For pricing Asian option with a payoff function, $V_T(S_1, S_2, \dots, S_p) = h((S_1 + S_2 + \dots + S_p)/p)$, for a function h , i.e. relying on the arithmetic average sampling values of asset price S_t at discrete time grids $t_i (t = 1, 2, \dots, p)$, or pricing a basket option with payoff function $h((S_1 + S_2 + \dots + S_p)/p)$, i.e. relying on arithmetic average values of several asset prices S_1, S_2, \dots, S_p at time T , we can take $X = \hat{V}_T(S_1, S_2, \dots, S_p) = h(\sqrt[p]{S_1 S_2 \dots S_p})$, i.e. relying on the geometric average of S_1, S_2, \dots, S_p , then by Equation (3),

$$0 \leq 1 - \rho_{XV}^2 \leq \frac{\epsilon_p(V_T, \hat{V}_T)}{\text{Var}[V_T]}, \quad (4)$$

where

$$\epsilon_p(V_T, \hat{V}_T) = E[(V_T(S_1, S_2, \dots, S_p) - \hat{V}_T(S_1, S_2, \dots, S_p))^2].$$

Thus, small $\epsilon_p(V_T, \hat{V}_T)$ means that X and V_T have high correlation.

Case (ii) For pricing a plain option with the payoff function $V_T = h(S_T)$, then we can take $X = h(\hat{S}_T)$, where virtual asset $\hat{S}_t (t \leq T)$ is an approximation of $S_t (t \leq T)$. Suppose that for some positive constant L , function h satisfies the condition

$$|h(x) - h(y)| \leq L|x - y|, \quad \forall x, y \in (-\infty, +\infty), \quad (5)$$

then we have $E[(V_T - X)^2] \leq L^2 E[(S_T - \hat{S}_T)^2]$. So by Equation (3), we have

$$0 \leq 1 - \rho_{XV}^2 \leq L^2 \frac{E[(S_T - \hat{S}_T)^2]}{\text{Var}[V_T]}, \quad (6)$$

which means that a good approximation of asset S_t , leads X and V_T to have high correlation.

Usually the error of estimator \bar{V} is measured by the following formula:

$$s[\bar{V}] = \sqrt{\frac{\sum_{j=1}^m (V_j - \bar{V})^2}{m-1}}.$$

In this paper we use the ratio of error

$$R = \frac{s[\bar{V}]}{s[\bar{V}(\hat{b})]}$$

to measure the improvement of the control variate method. It is close to $1/\sqrt{1 - \rho_{XV}^2}$ for large simulation paths m . We can get a great improvement if we construct a control variate with really high correlation with V .

We go through the following steps to price a derivative under stochastic volatility and the stochastic interest rate model by Monte Carlo simulation.

- *Step 1.* Generate some paths of the volatility and interest rate of the underlying asset.
- *Step 2.* Use the simulations in step 1 to construct a virtual asset with deterministic volatility and deterministic interest rate.
- *Step 3.* Choose a suitable derivative of the virtual asset as control variate. For example, discrete-sampled geometric average Asian option written on the virtual asset is often used as the control variate for pricing the original arithmetic average option, or use European option written on the virtual asset to calculate the price of the target European option.
- *Step 4.* Generate more paths of the underlying asset and virtual asset, calculate the value of $\bar{V}(b^*)$.

The key of the above processes is how to construct a suitable control variate X which has high correlation with V . We will discuss it in three typical situations in financial practice, i.e. the stochastic volatility model, stochastic interest rate model, and the hybrid model with stochastic volatility and stochastic interest rate, respectively.

2.2. Stochastic volatility model

In this subsection, we discuss how to construct a virtual asset which has similar movement and high correlation with the original asset. Many stochastic volatility models, such as the Hull–White model [15], Scott model [19], and the Heston model [14], have the following form:

$$\begin{aligned} dS_t &= r_t S_t dt + f(Y_t) S_t dW_{1t}, \\ dY_t &= \mu_Y(t, Y_t) dt + \sigma_Y(t, Y_t) dW_{2t}, \end{aligned} \quad 0 \leq t \leq T, \quad (7)$$

where r_t is a deterministic function of t , $f(Y)$, $\mu_Y(t, Y)$ and $\sigma_Y(t, Y)$ are given deterministic functions, W_{1t} and W_{2t} are standard Brown motions, $\text{Cov}(dW_{1t}, dW_{2t}) = \rho_{12} dt$.

If an underlying asset S_t satisfies Equation (7), we want to construct a virtual asset \hat{S}_t which has the following form:

$$d\hat{S}_t = r_t \hat{S}_t dt + \hat{\sigma}_t \hat{S}_t dW_{1t}$$

where r_t is the same deterministic function as in Equation (7), $\hat{\sigma}_t$ is a deterministic function which will be given later. We want to choose a suitable function $\hat{\sigma}_t$ to make sure that the virtual asset \hat{S}_t is close to the original asset S_t .

Euler discretization can be used to approximate paths of the asset on a discrete time grid set. Let $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ be a partition of a time interval $[0, T]$, for simplicity, we assume

that $t_i = i\Delta t$ for each $i = 0, 1, \dots, N$, and let $S(t_i) = S_i, \hat{S}(t_i) = \hat{S}_i, Y(t_i) = Y_i, \hat{\sigma}_t = \hat{\sigma}_i, r_{t_i} = r_i$, the discrete form for the asset process is given by

$$S_{i+1} = S_i e^{(r_i - (1/2)f^2(Y_i))\Delta t + f(Y_i)\sqrt{\Delta t}Z_{1,i}}, \tag{8}$$

where the process Y_i is also approximated by

$$Y_{i+1} = Y_i + \mu_Y(t_i, Y_i)\Delta t + \sigma_Y(t_i, Y_i)\sqrt{\Delta t}Z_{2,i} \tag{9}$$

and so the virtual asset process \hat{S}_i is chosen such a form

$$\hat{S}_{i+1} = \hat{S}_i e^{(r_i - (1/2)\hat{\sigma}_i^2)\Delta t + \hat{\sigma}_i\sqrt{\Delta t}Z_{1,i}}, \tag{10}$$

where $Z_{1,i}$ and $Z_{2,i}$ in Equations (8)–(10) are standard normal random variables with correlation coefficient $\text{Cov}(Z_{1,i}, Z_{2,i}) = \rho_{12}$. The common way to get samples of these two variables Z_1 and Z_2 is that generating independent standard normal random variables $Z_{1,i}$ and U_i and then letting $Z_{2,i} = \rho_{12}Z_{1,i} + \sqrt{1 - \rho_{12}^2}U_i$.

We want to choose a suitable function $\hat{\sigma}_t$ to make sure that \hat{S}_t close to the original asset S_t for all t . Equations (8)–(10) generate the paths of the original asset S_t and the virtual asset \hat{S}_t , it is easy to see that if all parameters in Equation (7) are given and the random variables $Z_{1,i}$ and $Z_{2,i}$ have been generated, the paths of S_t are determined, and the paths of \hat{S}_t are also determined by the values of $\hat{\sigma}_t$ at the discrete time grids $0 = t_0 < t_1 < t_2 < \dots < t_N = T$, i.e. by the vector $(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{N-1})$. We can construct a target function $F(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{N-1})$ to measure the difference between S_t and \hat{S}_t , and get the minimal point of F . In this way, we can make sure S_t and \hat{S}_t are close enough. The common way to measure the difference between S_t and \hat{S}_t is

$$\sum_{i=1}^N E[(S_i - \hat{S}_i)^2] \approx \frac{1}{m} \sum_{j=1}^m \sum_{i=1}^N (S_i^j - \hat{S}_i^j)^2,$$

where S_i^j and \hat{S}_i^j denote the j th path of S_t and \hat{S}_t at time $t = t_i$, generated by Equations (8)–(10), respectively. However, the functions above are very complicated to get the minimal points. The main reason is that $\hat{\sigma}_i$ is in the exponential position in the asset price formula (10). So for simplicity to deal in mathematics, we define the following target function instead:

$$\begin{aligned} F(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{N-1}) &= \sum_{i=0}^{N-1} E \left[\left(\ln \frac{S_{i+1}}{S_i} - \ln \frac{\hat{S}_{i+1}}{\hat{S}_i} \right)^2 \right] \\ &= \sum_{i=0}^{N-1} E \left[\left(\frac{1}{2}\hat{\sigma}_i^2 \Delta t - \hat{\sigma}_i\sqrt{\Delta t}Z_{1,i} - \frac{1}{2}f^2(Y_i)\Delta t + f(Y_i)\sqrt{\Delta t}Z_{1,i} \right)^2 \right]. \end{aligned}$$

There are some reasons to choose such a target function:

- First, S_t and \hat{S}_t become very close when the target function F closes to 0.
- Second, $\hat{\sigma}_i$ can be solved separately for $i = 0, 1, \dots, n - 1$, so the problem of determining values of $\hat{\sigma}_i$ becomes very simple even when the number of time steps becomes large.
- Third, the target function F is a polynomial for each variable $\hat{\sigma}_i$, so it is very easy to get the minimal point of F .

Next, we consider the following optimization problem:

$$\underset{\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{N-1}}{\text{minimize}} F(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{N-1}). \quad (11)$$

Theorem 2.1: Suppose that $\max_{0 \leq t \leq T} E[f^l(Y_t)] \leq \infty$, for $l = 1, 2$, $\Delta t \max_{0 \leq t \leq T} E[f^2(Y_t)] \leq 1$, then problem (11) possesses only one real solution $(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{N-1})$. Moreover, we have the following estimation for $i = 0, 1, \dots, N - 1$, respectively:

$$\hat{\sigma}_i = E[f(Y_i)]\{1 + \alpha \Delta t + \beta \Delta t^2 + O(\Delta t^3)\}, \quad (12)$$

where

$$\begin{aligned} \alpha &= \frac{1}{2} \text{Var}[f(Y_i)], \\ \beta &= \frac{1}{4} \text{Var}[f(Y_i)](E[f^2(Y_i)] - 3E^2[f(Y_i)]). \end{aligned}$$

Proof: Since $\lim_{\hat{\sigma}_0, \dots, \hat{\sigma}_{N-1} \rightarrow \infty} F(\hat{\sigma}_0, \dots, \hat{\sigma}_{N-1}) \rightarrow +\infty$, so the optimization (11) always exists solution, still denoted by notation $\hat{\sigma}_0, \dots, \hat{\sigma}_{N-1}$ for convenience. To prove Theorem 2.1, we first simplify the expression of F . To do this, it is realized from Equation (9) that for each $i = 1, 2, \dots, N$, Y_i only relies on Y_0 and random variables $Z_{2,0}, Z_{2,1}, \dots, Z_{2,i-1}$, and so it is independent with variable $Z_{1,i}$, which implies that

$$\begin{aligned} E[f(Y_i)Z_{1,i}] &= E[f(Y_i)]E[Z_{1,i}] = 0, \\ E[f^2(Y_i)Z_{1,i}] &= E[f^2(Y_i)]E[Z_{1,i}] = 0, \\ E[f(Y_i)Z_{1,i}^2] &= E[f(Y_i)]E[Z_{1,i}^2] = E[f(Y_i)]. \end{aligned}$$

Thus by calculation directly, we obtain

$$F(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{N-1}) = \sum_{i=0}^{N-1} \left\{ \frac{1}{4} \Delta t^2 \hat{\sigma}_i^4 + \left(\Delta t - \frac{1}{2} \Delta t^2 E[f^2(Y_i)] \right) \hat{\sigma}_i^2 - 2\Delta t E[f(Y_i)] \hat{\sigma}_i + F_0 \right\}, \quad (13)$$

where $F_0 = E[(-\frac{1}{2}f^2(Y_i)\Delta t + f(Y_i)\sqrt{\Delta t}Z_{1,i})^2]$ is a constant.

Next, by taking the first- and two-order derivatives of the function F , and let ψ be a function of one variable $\hat{\sigma}_i$,

$$\psi(\hat{\sigma}_i) = \frac{\partial F}{\partial \hat{\sigma}_i},$$

then we have that

$$\psi(\hat{\sigma}_i) = \Delta t^2 \hat{\sigma}_i^3 + \Delta t(2 - \Delta t E[f^2(Y_i)])\hat{\sigma}_i - 2\Delta t E[f(Y_i)], \quad (14)$$

$$\frac{\partial \psi}{\partial \hat{\sigma}_i}(\hat{\sigma}_i) = 3\Delta t^2 \hat{\sigma}_i^2 + \Delta t(2 - \Delta t E[f^2(Y_i)]). \quad (15)$$

Let

$$\begin{aligned} \lambda^* &= E[f(Y_i)](1 + \Delta t E[f^2(Y_i)]), \\ \lambda_* &= E[f(Y_i)](1 - \Delta t E^2[f(Y_i)]), \end{aligned}$$

then if $\Delta t \max_{0 \leq t \leq T} E[f^2(Y_t)] \leq 1$, we have from Equation (14) that

$$\begin{aligned} \psi(\lambda^*) &= \Delta t^2 E^3[f(Y_i)](1 + \Delta t E[f^2(Y_i)])^3 + \Delta t^2 E[f^2(Y_i)](1 - \Delta t E[f^2(Y_i)]) \\ &> \Delta t^2 E[f^2(Y_i)](1 - \Delta t E[f^2(Y_i)]) \\ &\geq \Delta t^2 E[f^2(Y_i)] \left(1 - \Delta t \max_{0 \leq t \leq T} E[f^2(Y_t)] \right) \\ &\geq 0, \end{aligned} \tag{16}$$

$$\begin{aligned} \psi(\lambda_*) &= \Delta t^2 E^3[f(Y_i)](1 - \Delta t E[f^2(Y_i)])^3 \\ &\quad - \Delta t^2 E[f(Y_i)]\{E[f^2(Y_i)] + 2E^2[f(Y_i)] - \Delta t E[f^2(Y_i)]E^2[f(Y_i)]\} \\ &< \Delta t^2 E^3[f(Y_i)] - \Delta t^2 E[f(Y_i)]\{E[f^2(Y_i)] + 2E^2[f(Y_i)] - \Delta t E[f^2(Y_i)]E[f(Y_i)]\} \\ &= \Delta t^2 E[f(Y_i)]E[f^2(Y_i)](\Delta t E[f^2(Y_i)] - 1) - \Delta t^2 E^3[f^2(Y_i)] \\ &\leq \Delta t^2 E[f(Y_i)]E[f^2(Y_i)] \left(\Delta t \max_{0 \leq t \leq T} E[f^2(Y_t)] - 1 \right) \\ &\leq 0. \end{aligned} \tag{17}$$

On the other hand, if $\Delta t \max_{0 \leq t \leq T} E[f^2(Y_t)] \leq 1$, we have from Equation (15) that

$$\begin{aligned} \frac{\partial \psi}{\partial \hat{\sigma}_i}(\hat{\sigma}_i) &= 3\Delta t^2 \hat{\sigma}_i^2 + \Delta t(2 - \Delta t E[f^2(Y_i)]) \\ &\geq \Delta t \left(2 - \max_{0 \leq t \leq T} \Delta t E[f^2(Y_t)] \right) \\ &> 0. \end{aligned} \tag{18}$$

The combination of results (15)–(17) leads to the result that equation $\psi(\hat{\sigma}_i) = 0$ has and only has one real solution $\hat{\sigma}_i$ in the interval $[0, \infty)$. Furthermore $\hat{\sigma}_i \in (\lambda_*, \lambda^*)$. By Equation (14), it is clear that equation $\psi(\hat{\sigma}_i) = 0$ is equivalent to

$$\hat{\sigma}_i = \frac{E[f(Y_i)]}{1 + \frac{1}{2}\Delta t\{\hat{\sigma}_i^2 - E[f^2(Y_i)]\}},$$

we finally finish the proof of Theorem 2.1 immediately by taking a recursion procedure to the above formula. ■

Remark 2.2: In practice, we can calculate $\hat{\sigma}_i (i = 0, 1, \dots, N)$ by simulations, i.e. by Equation (12), for small Δt and suitable large m ,

$$\hat{\sigma}_i \approx E[f(Y_i)] \approx \frac{1}{m} \sum_{j=1}^m f(Y_i^j), \tag{19}$$

where Y_i^j is the j -path value of variable Y_t at time $t = t_i$, calculated by recursion formula (9).

Compared with the usual numerical method to solve the optimization problem for function F , the approximate formula (12) saves a lot of computing time even for large number of N . We will give an

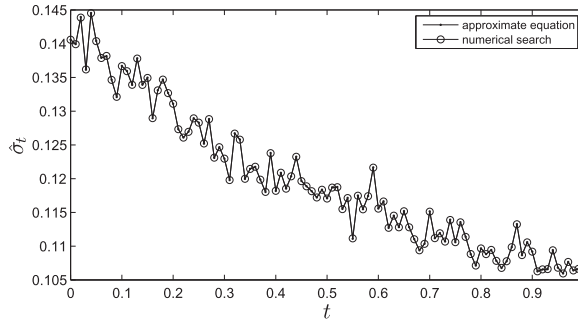


Figure 1. Optimal volatilities by numerical search and the approximate formula (19). Where ‘ \cdot ’ is the result from the approximate equation, and ‘ \circ ’ is the result from numerical search.

example to show that even using the formula (19), there is also a satisfactory computational accuracy. We assume that the underlying asset satisfies the Heston model

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{Y_t}S_t dW_{1t}, \\ dY_t &= a_Y(\theta_Y - Y_t) dt + \sigma_Y\sqrt{Y_t} dW_{2t}, \end{aligned} \quad (20)$$

where $r = 5\%$, $a_Y = 2$, $\theta_Y = 0.01$, $\sigma_Y = 0.02$, $T = 1$, $Y_0 = 0.02$, $S_0 = 100$, $\rho_{12} = 0.5$. The virtual asset \hat{S}_t has the form

$$d\hat{S}_t = r\hat{S}_t dt + \hat{\sigma}_t\hat{S}_t dW_{1t}.$$

The partition of time is $0 = t_0 < t_1 < t_2 < \dots < t_{100} = 1$, simulation times $m = 300$. The volatility of the virtual asset $\hat{\sigma}_t$ can be represented by a vector $(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{99})$, which is the minimal point of the target function F , we get it by two methods, the first one is by the search method for solving minimal point of F , and the second one is by the simple approximate formula (19). Figure 1 shows that the two results are almost totally the same, so we will use Equation (19) to get $(\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_{N-1})$ in the sequel.

The estimation result in Equation (12) means that our result in this subsection is similar to the corresponding one in [11] and Theorem 3.2 in [17] in a special case with $f(Y_t) = \sqrt{Y_t}$, $\mu_Y(t, Y_t) = \mu Y_t$ and $\sigma_Y(t, Y_t) = \hat{\sigma} Y_t$ for some constants $\mu, \hat{\sigma}$. However, the method used here has little restriction on analytic form of functions $f(Y_t)$, $\mu_Y(t, Y)$ and $\sigma_Y(t, Y)$ in model (7).

We now give a numerical example for the Heston model (20) to show that the virtual asset \hat{S}_t does have really high correlation with S_t , and keep all parameters the same as above. Figure 2 plots 300 dots (S_T^j, \hat{S}_T^j) , $j = 1, 2, \dots, 300$, for $N = 100$. We can see that S_T and \hat{S}_T have very high correlation from the figure. In fact, not only on the time T , but also on all time $0 < t < T$, S_t and \hat{S}_t have high correlation. So it is reasonable to believe that taking the corresponding derivatives written on the virtual assets is a good control variate when pricing derivative written on asset S_t , even for the path-dependent options. We will show this conclusion by several typical numerical examples in Section 3.

2.3. Stochastic interest rate model

In this subsection, we will construct a virtual asset which has a similar movement as the asset driven by a stochastic interest rate model. Consider a stochastic interest rate model with the following form (see e.g. [2]):

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma_t S_t dW_{1t}, \\ dr_t &= \mu_r(t, r_t) dt + \sigma_r(t, r_t) dW_{2t}, \end{aligned} \quad 0 \leq t \leq T, \quad (21)$$

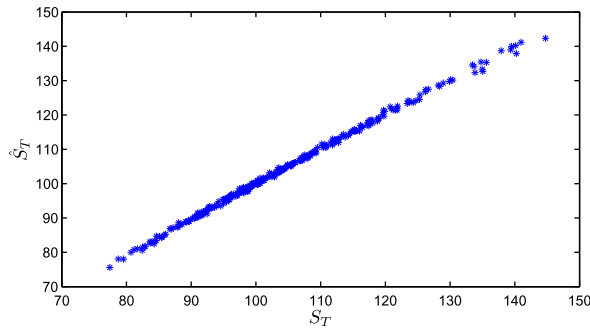


Figure 2. Figure of S_T and \hat{S}_T . The correlation coefficient of S_T and \hat{S}_T is 0.9992.

where $\sigma_r(t, r)$, $\mu_r(t, r)$ and σ_t are deterministic functions, $W_{1,t}$ and $W_{2,t}$ are standard Brown motions, $\text{Cov}(dW_{1t}, dW_{2t}) = \rho_{12} dt$. We shall construct a virtual asset \hat{S}_t which has the following form:

$$d\hat{S}_t = \hat{r}_t \hat{S}_t dt + \sigma_t \hat{S}_t dW_{1t}, \tag{22}$$

where function σ_t and Brown movement W_{1t} are the same as in Equation (21), \hat{r}_t is a deterministic function which will be chosen such that the virtual asset \hat{S}_t closes to the original one S_t .

Let $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ be a partition of time with N equal segments of length Δt , and let $S(t_i) = S_i$, $\hat{S}(t_i) = \hat{S}_i$, $\hat{r}_i = \hat{r}_i$, $\sigma_{t_i} = \sigma_i$, the discretization form of the asset process S_t is

$$S_{i+1} = S_i e^{(r_i - (1/2)\sigma_i^2)\Delta t + \sigma_i \sqrt{\Delta t} Z_{1,i}}, \tag{23}$$

the interest rate process r_t is discretized by

$$r_{i+1} = r_i + \mu_r(t_i, r_i)\Delta t + \sigma_r(t_i, r_i)\sqrt{\Delta t} Z_{2,i}, \tag{24}$$

and so the virtual asset process is chosen as follows:

$$\hat{S}_{i+1} = \hat{S}_i e^{(\hat{r}_i - (1/2)\sigma_i^2)\Delta t + \sigma_i \sqrt{\Delta t} Z_{1,i}}, \tag{25}$$

where $Z_{1,i}$ and $Z_{2,i}$ are standard normal random variables with $\text{Cov}(Z_{1,i}, Z_{2,i}) = \rho_{12}$.

Suppose that $\max_{0 \leq t \leq T} E[|r^l(t)|] \leq \infty$, for $l = 1, 2$. In order to choose a suitable set of \hat{r}_t , We use a similar method as in the previous subsection. Define a target function

$$\begin{aligned} F(\hat{r}_0, \hat{r}_1, \dots, \hat{r}_{N-1}) &= \sum_{i=0}^{N-1} E \left[\left(\ln \frac{S_{i+1}}{S_i} - \ln \frac{\hat{S}_{i+1}}{\hat{S}_i} \right)^2 \right] \\ &= \sum_{i=0}^{N-1} E[(r_i - \hat{r}_i)^2 \Delta t^2]. \end{aligned} \tag{26}$$

It is easy to see that the minimal point $(\hat{r}_0, \hat{r}_1, \dots, \hat{r}_{N-1})$ of F satisfies

$$\hat{r}_i = E[r_i], \quad i = 0, 1, \dots, N - 1. \tag{27}$$

Then we can construct the virtual asset \hat{S}_t by formula (25).

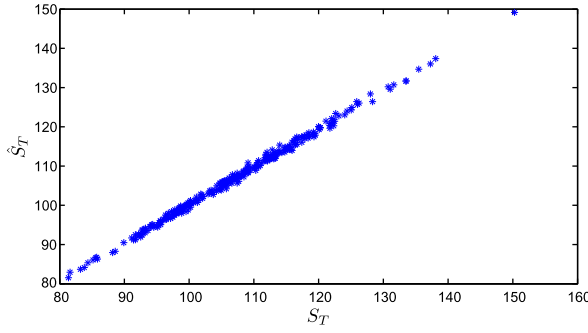


Figure 3. Figure of S_T and \hat{S}_T . The correlation coefficient of S_T and \hat{S}_T is 0.9981.

Remark 2.3: In practice, for suitable large m , we can see from Equation (27)

$$\hat{r}_i \approx \frac{1}{m} \sum_{j=1}^m r_i^j, \quad (28)$$

for $i = 0, 1, \dots, N-1$, where r_i^j is the j th path of variable r_t at time $t = t_i$ generated by formula (24).

We still use an example to show the high correlation between the virtual asset \hat{S}_t and the original asset S_t . Assume the underlying asset satisfies the CIR interest rate model,

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma S_t dW_{1t}, \\ dr_t &= a_r(\theta_r - r_t) dt + \sigma_r \sqrt{r_t} dW_{2t}, \end{aligned} \quad 0 \leq t \leq T,$$

where $T = 1$, $\sigma = 0.1$, $a_r = 2$, $\theta_r = 0.05$, $\sigma_r = 0.1$, $r_0 = 0.05$, $S(0) = 100$, $\rho_{12} = 0.5$.

We generate 300 paths of S_t and \hat{S}_t , Figure 3 plots the relation between S_T and \hat{S}_T with $N = 100$, which also shows the high correlation between \hat{S}_T and S_T .

2.4. Hybrid model with stochastic volatility and stochastic interest rate

This subsection considers a more complicated model of asset price, we assume that the underlying asset satisfies the following hybrid model with stochastic volatility and stochastic interest rate simultaneously:

$$\begin{aligned} dS_t &= r_t S_t dt + S_t f(Y_t) dW_{1t}, \\ dr_t &= \mu_r(t, r_t) dt + \sigma_r(t, r_t) dW_{2t}, \\ dY_t &= \mu_Y(t, Y_t) dt + \sigma_Y(t, Y_t) dW_{3t}, \end{aligned} \quad 0 \leq t \leq T, \quad (29)$$

where $\mu_r(t, r)$, $\sigma_r(t, r)$, $\mu_Y(t, Y)$, $\sigma_Y(t, Y)$ and $f(Y)$ are given deterministic functions, W_{1t} , W_{2t} and W_{3t} are standard Brown motions with $\text{Cov}(dW_{1t}, dW_{2t}) = \rho_{12} dt$, $\text{Cov}(dW_{1t}, dW_{3t}) = \rho_{13} dt$, $\text{Cov}(dW_{2t}, dW_{3t}) = \rho_{23} dt$.

Let $0 = t_0 < t_1 < t_2 < \dots < t_N = T$ be a partition of time with N equal segments of length Δt , and let $S(t_i) = S_i (i = 0, 1, \dots, N)$ be the discretization form of the asset process,

$$S_{i+1} = S_i e^{(r_i - (1/2)f^2(Y_i))\Delta t + f(Y_i)\sqrt{\Delta t}Z_{1,i}}, \quad (30)$$

where r_i and Y_i are discretized by

$$\begin{aligned} r_{i+1} &= r_i + \mu_r(t_i, r_i)\Delta t + \sigma_r(t_i, r_i)\sqrt{\Delta t}Z_{2,i}, \\ Y_{i+1} &= Y_i + \mu_Y(t_i, Y_i)\Delta t + \sigma_Y(t_i, Y_i)\sqrt{\Delta t}Z_{3,i}. \end{aligned} \quad (31)$$

As in the previous subsections, we construct a virtual asset by the following model:

$$\hat{S}_{i+1} = \hat{S}_i e^{(\hat{r}_i - (1/2)\hat{\sigma}_i^2)\Delta t + \hat{\sigma}_i\sqrt{\Delta t}Z_{1,i}}, \quad (32)$$

where \hat{r}_i and $\hat{\sigma}_i$ are some parameters, which will be given later.

Define a function F ,

$$\begin{aligned} F(\hat{r}_0, \dots, \hat{r}_{N-1}, \hat{\sigma}_0, \dots, \hat{\sigma}_{N-1}) &= \sum_{i=0}^{N-1} E \left[\left(\ln \frac{S_{i+1}}{S_i} - \ln \frac{\hat{S}_{i+1}}{\hat{S}_i} \right)^2 \right] \\ &= \sum_{i=0}^{N-1} E \left[\left(\left(\frac{1}{2}\hat{\sigma}_i^2 - \hat{r}_i \right) \Delta t - \hat{\sigma}_i\sqrt{\Delta t}Z_{1,i} + \left(r_i - \frac{1}{2}f^2(Y_i) \right) \Delta t + f(Y_i)\sqrt{\Delta t}Z_{1,i} \right)^2 \right], \end{aligned}$$

we need to find $(\hat{r}_0, \dots, \hat{r}_{N-1}, \hat{\sigma}_0, \dots, \hat{\sigma}_{N-1})$, such that

$$(\hat{r}_0, \dots, \hat{r}_{N-1}, \hat{\sigma}_0, \dots, \hat{\sigma}_{N-1}) = \operatorname{argmin} F(\hat{r}_0, \dots, \hat{r}_{N-1}, \hat{\sigma}_0, \dots, \hat{\sigma}_{N-1}). \quad (33)$$

Theorem 2.4: Suppose that $\max_{0 \leq t \leq T} E[f^l(Y_t)] \leq \infty$, $\max_{0 \leq t \leq T} E[|r^l(t)|] \leq \infty$ for $l = 1, 2$. Then the solution of problem (33) satisfies

$$\begin{aligned} \hat{r}_i &= E[r_i] - \frac{1}{2} \operatorname{Var}[f(Y_i)], \\ \hat{\sigma}_i &= E[f(Y_i)] \end{aligned} \quad (34)$$

for $i = 0, 1, \dots, N - 1$.

Proof: In a similar method as in the derivation of Equation (13), we find that

$$\begin{aligned} F(\hat{r}_0, \dots, \hat{r}_{N-1}, \hat{\sigma}_0, \dots, \hat{\sigma}_{N-1}) &= \sum_{j=1}^{N-1} \left\{ \left(\hat{r}_i - \frac{1}{2}\hat{\sigma}_i^2 \right)^2 \Delta t^2 - 2 \left(\hat{r}_i - \frac{1}{2}\hat{\sigma}_i^2 \right) \Delta t^2 E \left[r_i - \frac{1}{2}f^2(Y_i) \right] \right. \\ &\quad \left. + \hat{\sigma}_i^2 \Delta t - 2\hat{\sigma}_i \Delta t E[f(Y_i)] + F_0 \right\}, \end{aligned}$$

where $F_0 = E[(r_i - \frac{1}{2}f^2(Y_i))\Delta t + f(Y_i)\sqrt{\Delta t}Z_{1,i}]^2$ is a constant.

Let $\partial F/\partial \hat{r}_i = 0$, $\partial F/\partial \hat{\sigma}_i = 0$, then we have that

$$\hat{r}_i = \frac{1}{2}\hat{\sigma}_i^2 + E[r_i] - \frac{1}{2}E[f^2(Y_i)], \quad (35)$$

$$- \left(\hat{r}_i - \frac{1}{2}\hat{\sigma}_i^2 \right) \hat{\sigma}_i \Delta t^2 + \hat{\sigma}_i \Delta t^2 E \left[r_i - \frac{1}{2}f^2(Y_i) \right] + 2\hat{\sigma}_i \Delta t - 2\Delta t E[f(Y_i)] = 0. \quad (36)$$

Inserting Equation (35) into (36) implies $\hat{\sigma}_i = E[f(Y_i)]$, thus $\hat{r}_i = E[r_i] - \frac{1}{2} \operatorname{Var}[f(Y_i)]$, and we complete the proof of Theorem 2.4. ■

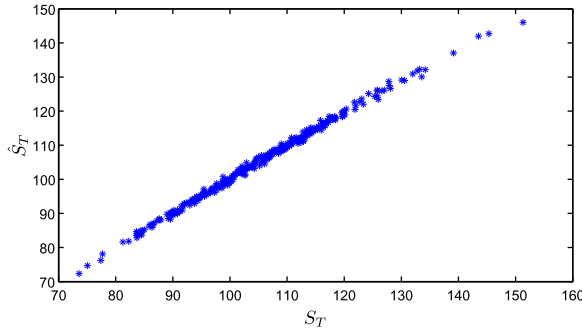


Figure 4. Figure of S_T and \hat{S}_T . The correlation coefficient of S_T and \hat{S}_T is 0.9978.

Remark 2.5: In practice, we can calculate $\hat{\sigma}_i (i = 0, 1, \dots, N - 1)$ by simulations, i.e. by Equation (34), for suitable large m ,

$$\hat{r}_i \approx \frac{1}{m} \sum_{j=1}^m r_i^j - \frac{1}{2} \left[\frac{1}{m} \sum_{j=1}^m f^2(Y_i^j) - \left(\frac{1}{m} \sum_{j=1}^m f(Y_i^j) \right)^2 \right], \quad (37)$$

$$\hat{\sigma}_i \approx \frac{1}{m} \sum_{j=1}^m f(Y_i^j)$$

where r_i^j and Y_i^j are the j th path of variables r_t and Y_t at time $t = t_i$ generated by formula (31).

We still use an example to show the high correlation between S_t and \hat{S}_t . Let the asset S_t satisfy the following Heston-CIR model:

$$\begin{aligned} dS_t &= r_t S_t dt + \sqrt{Y_t} S_t dW_{1t}, \\ dr_t &= a_r(\theta_r - r_t) dt + \sigma_r \sqrt{r_t} dW_{2t}, \quad 0 \leq t \leq T, \\ dY_t &= a_Y(\theta_Y - Y_t) dt + \sigma_Y \sqrt{Y_t} dW_{3t}, \end{aligned}$$

where $a_Y = a_r = 2$, $\theta_Y = 0.01$, $\theta_r = 0.05$, $\sigma_Y = 0.02$, $\sigma_r = 0.1$, $T = 1$, $Y_0 = 0.02$, $r_0 = 5\%$, $S_0 = 100$, $\text{Cov}(dW_{1t}, dW_{2t}) = 0.5dt$, $\text{Cov}(dW_{1t}, dW_{3t}) = 0.5dt$, $\text{Cov}(dW_{2t}, dW_{3t}) = 0.5dt$.

We generate 300 paths of S_t and \hat{S}_t , Figure 4 plots the relation between S_T and \hat{S}_T with $N = 100$, which shows \hat{S}_T has very high correlation with S_T .

3. Numerical examples

In this section, we will give several examples of option pricing under different asset models to test how our method works.

3.1. Asian option with the Heston model

In this subsection, we will give an example of pricing an Asian option with the Heston stochastic volatility model. Assume the underlying asset S_t is determined by

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{Y_t} S_t dW_{1t}, \\ dY_t &= a_Y(\theta_Y - Y_t) dt + \sigma_Y \sqrt{Y_t} dW_{2t}. \end{aligned} \quad 0 \leq t \leq T,$$

Table 1. Prices and errors of the Asian option estimated by CMC and CVMC method. Simulation paths $m = 10^4$.

Strike price	CMC price (Error)	CVMC price (Error)	Ratio of error R
95	7.5897 (0.0567)	7.6861 (0.0025)	22.54
100	3.9192 (0.0460)	3.9432 (0.0021)	21.41
105	1.5644 (0.0306)	1.5680 (0.0017)	17.60

Taking coefficients $r = 5\%$, $a_Y = 2$, $\theta_Y = 0.01$, $\sigma_Y = 0.02$, $T = 1$, $Y_0 = 0.01$, $S_0 = 100$, $\text{Cov}(dW_{1t}, dW_{2t}) = 0.5dt$, and the observed dates for discrete-sampled Asian option are $T_i = i/10$, $i = 1, 2, \dots, 10$. Then the payoff of Asian option is

$$\left(\frac{1}{10} \sum_{i=1}^{10} S_{T_i} - K \right)^+.$$

We first construct a virtual asset \hat{S}_t by the formula (4) in Section 2, and then choose the geometric average option written on \hat{S} as a control variate, as suggested by Kemna and Vorst [16], which has an analytical formula (see [11, p.99] or [16]). The price of geometric average call option is

$$C = e^{-\int_0^T r dt} E[(G(T) - K)^+],$$

where

$$G(T) = \left(\prod_{i=1}^n \hat{S}_{T_i} \right)^{1/n}.$$

Since

$$\hat{S}_{T_i} = S_0 e^{\int_0^{T_i} (r - \hat{\sigma}_t^2/2) dt + \int_0^{T_i} \hat{\sigma}_t dW_{1t}},$$

it is obvious that $\ln G(T)$ is normally distributed with expectation

$$E^* = \ln S_0 + \frac{1}{n} \sum_{i=1}^n \int_0^{T_i} (r - \hat{\sigma}_t^2/2) dt$$

and variance

$$V^* = \sum_{i=1}^n \int_{T_{i-1}}^{T_i} \left(\frac{n+1-i}{n} \hat{\sigma}_t \right)^2 dt.$$

Therefore, the value of the geometric average call option can be presented as follows:

$$C = e^{-rT} \left(e^{E^* + V^*/2} N \left(\frac{E^* - \ln K + V^*}{\sqrt{V^*}} \right) - KN \left(\frac{E^* - \ln K}{\sqrt{V^*}} \right) \right).$$

The prices and errors of simulation for the Asian option estimated by classical Monte Carlo (CMC) and the Monte Carlo method with control variate (CVMC) and ratios of errors for various values of strike price K are listed in Table 1, where

$$R = \frac{\sqrt{\text{Var}_{\text{CMC}}}}{\sqrt{\text{Var}_{\text{CVMC}}}}$$

is the ratio of standard deviations.

Table 2. Errors of the Asian option price estimated by CMC and CVMC methods for various σ_Y , simulation paths $m = 10^4$, strike price $K = 100$.

σ_Y	Error using CMC	Error using CVMC	Ratio of error R
0.02	0.0460	0.0021	21.41
0.05	0.0470	0.0045	10.40
0.1	0.0478	0.0090	5.32

Table 3. Errors of Asian option price estimated by CMC and CVMC methods for various values of ρ_{12} , simulation paths $m = 10^4$, $\sigma_Y = 0.02$.

ρ_{12}	Error using CMC	Error using CVMC	Ratio of error R
0.8	0.0464	0.0021	22.06
0	0.0454	0.0022	21.10
-0.8	0.0455	0.0020	22.75

Table 4. Results by choosing different control estimators for $\rho_{12} = 0.8$.

T	R_1	R_2	R_3
$T = 1$	22.06	12.31	20.27
$T = 5$	18.73	5.03	16.35

In the Heston model, the volatility of volatility σ_Y is a very important parameter, we keep all other parameters to be the same as the example above, and just change σ_Y to see how our method works in different situations. The Feller condition means $0 < \sigma_Y < \sqrt{2a\theta_Y} = 0.2$, so we set σ_Y be 0.02, 0.05, and 0.1, respectively, Table 2 shows the ratios of error reduce when σ_Y increases. The reason is that the underlying asset becomes hard to control when the volatility of volatility increases, so the effect of control variate method becomes more weak. But even though σ_Y increases to 0.1, which is a large number in practice, we still get a good result, the error is reduced from 0.0478 to 0.0090.

Table 3 presents the ratios of error for various values of ρ_{12} . which shows that our method still works well in different situations.

In order to compare the advantages of our method and the previous results for different T , we take options written on virtual assets with constant volatility, as suggested by Glasserman [11], time-varying function as suggested by Ma and Xu [17] as controlled estimators, respectively. Numerical results in Table 4 show the efficiency of our method even for $T = 5$. $R_i (i = 1, 2, 3)$ represents the ratio of simulation errors as above by taking formula (19), $\rho_{12} = 0.8$, $Y_0 = 0.015$, $r = 5\%$, $\theta_Y = 0.01$, $\alpha_Y = 2$, $\hat{\sigma}_t^2 = Y_0 = 0.02$ and $\hat{\sigma}_t^2 = \theta_Y - (\theta_Y - Y_0)e^{-\alpha_Y t}$, respectively.

3.2. Asian option and European option with the HCIR model

Assume the underlying asset S_t is driven by the following HCIR model (the Heston stochastic volatility model with CIR interest rate):

$$\begin{aligned} dS_t &= r_t S_t dt + \sqrt{Y_t} S_t dW_{1t}, \\ dr_t &= a_r(\theta_r - r_t) dt + \sigma_r \sqrt{r_t} dW_{2t}, \quad 0 \leq t \leq T, \\ dY_t &= a_Y(\theta_Y - Y_t) dt + \sigma_Y \sqrt{Y_t} dW_{3t}, \end{aligned}$$

where $a_Y = a_r = 2$, $\theta_Y = 0.01$, $\theta_r = 5\%$, $T = 1$, $Y_0 = 0.01$, $r_0 = 5\%$, $S_0 = 100$, $\text{Cov}(dW_{1t}, dW_{2t}) = 0.5dt$, $\text{Cov}(dW_{1t}, dW_{3t}) = 0.5dt$, $\text{Cov}(dW_{2t}, dW_{3t}) = 0$, σ_r and σ_Y will be determined later.

Table 5. Errors of Asian option prices estimated by CMC and CVMC methods for various values of σ_Y and σ_r .

σ_Y	σ_r	Error using CMC	Error using CVMC	Ratio of error
0.02	0.05	0.0469	0.0026	18.20
0.02	0.1	0.0473	0.0032	14.76
0.02	0.2	0.0480	0.0052	9.26
0.05	0.05	0.0473	0.0049	9.62
0.05	0.1	0.0487	0.0055	8.89
0.05	0.2	0.0490	0.0069	7.11
0.1	0.05	0.0487	0.0094	5.23
0.1	0.1	0.0484	0.0095	5.10
0.1	0.2	0.0501	0.0106	4.71

Note: Simulation paths $m = 10^4$.

Table 6. Errors of European option estimated by CMC and CVMC with different values of σ_Y and σ_r , taking simulation paths $m = 10^4$.

σ_Y	σ_r	Error using CMC	Error using CVMC	Ratio of error
0.02	0.05	0.0790	0.0048	16.36
0.02	0.1	0.0803	0.0067	12.02
0.05	0.05	0.0818	0.0099	8.29
0.05	0.1	0.0829	0.0109	7.58

Table 7. Errors of European option estimated by CMC and CVMC with different values of r_0 and Y_0 .

r_0	Y_0	Error using CMC	Error using CVMC	Ratio of error
0.05	0.1	0.0829	0.0109	7.58
0.02	0.1	0.0603	0.0047	14.02
0.07	0.1	0.0468	0.0027	18.06
0.05	0.05	0.0370	0.00219	17.38

Note: Taking simulation paths $m = 10^4$.

3.2.1. Asian option

Consider an Asian option, we still use geometric average option written on the virtual asset as a control variate which is described in Section 3.1. From examples under the Heston stochastic volatility model in the above subsection, we know that the most important parameters that affect the ratio of error are σ_Y and σ_r . Table 5 shows the ratios of error with different values of σ_Y and σ_r under the HCIR model. The ratios of error are still acceptable when σ_Y and σ_r become large.

3.2.2. European option

Consider a European call option. We use the corresponding European call option written on the virtual asset as a control variate, whose price can be calculate by the famous Black-Scholes formula. Table 6 shows the ratios of error with strick price $K = 100$ and different values of σ_Y and σ_r .

For testing the effect of our method for different values of r_0 and Y_0 , we list some numerical results in Table 7, where $a_Y = a_r = 2$, $\theta_Y = 0.01$, $\theta_r = 5\%$, $T = 1$, $\sigma_r = 0.1$, $\sigma_Y = 0.05$, $S_0 = 100$, $\text{Cov}(dW_{1t}, dW_{2t}) = 0.5dt$, $\text{Cov}(dW_{1t}, dW_{3t}) = 0.5dt$, $\text{Cov}(dW_{2t}, dW_{3t}) = 0$.

4. Conclusions

This paper tries to improve the efficiency of Monte Carlo simulation in pricing derivatives under stochastic volatility and the stochastic interest rate model by the control variate method. If the underlying asset is driven by a complicated model, the common control variates may not be easy to construct because the control variates may not have a closed-form expectation in this situation. We construct

a virtual asset with deterministic volatility and interest rate. Since it possesses piecewise constant volatility and interest rate value functions, the value of the corresponding derivative written on the virtual asset is easy to know. Numerical results in the paper show the efficiency of our method. Some theoretic results can help us to understand the mechanism of a control variate.

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